The Journal of Vational Chengchi University Vol. 36, 1977

ON A CONJECTURE OF ADAMS

WEN-HSIUNG LIN

Associate Professor

Department of Applied Mathematics

National Chengchi University

1. INTRODUCTION

In [2] J.F. Adams poses the following conjecture concerning the stable cohomotopy groups of the infinite real projective space RP^∞.

Conjecture A The group of maps in Boardman's category from the suspension spectrum of RP^∞ to the suspension spectrum of S^n is zero if $n > 0$.

The purpose of this paper is to study this conjecture from another conjecture of Adams (also in [2]) on the vanishing of Ext groups of a certain module over the mod 2 Steenrod algebra A. We show that the latter conjecture implies Conjecture A.

In order to state the latter conjecture let $Z_2[x,x^{-1}]$ be the ring of finite Laurent series over Z_2 with $\dim(x)=1$. $Z_2[x,x^{-1}]$ is a left A-module with A-action given by

$$Sq^i x^k = \binom{k}{i} x^{i+k} \qquad k \geq 0$$

$$Sq^i x^{-k} = \binom{2^m-k}{i} x^{i-k} \qquad k \geq 1$$

where m is large compared with k and i. $Z_2[x,x^{-1}]$ is considered in [2] as the cohomology of a hypothetical spectrum which is like the suspension spectrum of RP^∞, but has one cell in each dimension p whether p is positive, negative or zero. We refer to [2] for more story on $Z_2[x,x^{-1}]$ and why the following conjecture was put forth. Let M be the Z_2-submodule of $Z_2[x,x^{-1}]$generated by all x^i with $i\neq-1$. It is easy to see that M is an A-submodule of $Z_2[x,x^{-1}]$.

Conjecture B $\mathrm{Ext}_A^{s,t}(M,Z_2)=0$ for all s and t.

The main result of the paper is

Theorem 1.1 If Conjecture B is true then so is Conjecture A.

Theorem 1.1 seems known to experts in the field, but it appears that no detail proofs are present in the literature. It is the purpose the paper to record a detail proof of Theorem 1.1. which, in expert's mind, might be nonstandard.

We sall use the Adams spectral sequence arguments to prove Theorem 1.1. Let $E(RP^\infty)$ denote the suspension spectrum of RP^∞, S^0 the sphere spectrum and let $[E(RP^\infty),S^0]_r$ be the group of maps of degree r from $E(RP^\infty)$to S^0 in the Boardman stable category S_h ([1],[3]). If we try to prove Conjecture A , by using the Adams spectral sequence arguments, we need to show that (1) in the Adams spectral squence $\{E_r[E(RP^\infty),S^0]\}$, $E_\infty^{s,t}=0$ if $t-s<0$ and (2) the spectral sequence $\{E_r(E(RP^\infty),S^0)\}$ converges to $[E(RP^\infty),S^0]_*$ at least in dimensions of interest.

Theorem 1.2 Suppose Conjecture B is true. Then $E_2^{s,t}(E(RP^\infty),S^0)=\mathrm{Ext}_A^{s,t}(Z_2, H^*(RP^\infty))$ (and hence $E_\infty^{s,t})=0$ if $t-s<0$ where $H^*(RP^\infty)$ is the reduced mod 2 cohomology of RP^∞.

So to prove Theorem 1.1 it remains to consider the convergence of the Adams spectral sequence $\{E_r[E(RP^\infty),S^0]\}$. Consider the following mod 2 Adams resolution of S^0 in S_h corresponding to the minimal resolution of Z_2 over A.

$$\begin{array}{ccccc} & & \vdots & & \\ S^{-1}K_1 & \longrightarrow & Y_2 & \longrightarrow & K_2 \\ & & \downarrow & & \\ S^{-1}K_0 & \longrightarrow & Y_1 & \longrightarrow & K_1 \\ & & \downarrow & & \\ & & Y_0=S^0 & \longrightarrow & K_0 \end{array}$$

Let Y_∞ be the homotopy inverse limit spectrum of the sequence
$\cdots \to Y_2 \to Y_1 \to Y_0 = S^0$.

Theorem 1.3 In S_h $[E(RP^\infty), Y_\infty]_r = 0$ all r.

It is easy to see that Theorem 1.1 follow from 1.2 and Theorem 1.3.

2. Proof of Theorem 1.2

Recall that $H^*(RP^\infty) = Z_2[u]$ where dim $(u)=1$. So $H^*(RP^\infty)$ is a (graded) vector space over Z_2 with base $\{u, u^2, u^3, \cdots\}$. We reindex the Z_2-dual module $[H^*(RP^\infty)]^* = H_*(RP^\infty)$ (i.e. the reduced mod 2 homology of RP^∞) by assigning to u^{k*} the degree -k. Write $D(H^*(RP^\infty))$ for the reindexen $H_*(RP^\infty)$ aud y_{-k} for u^{k*}. Define a left A-action on $D[H^*(RP^\infty)]$ by $ay_{-k} = y_{-k}\chi(a)$ where χ: A $\longrightarrow$ A is the canonical antiautomorphism of the Steenrod algebra A and the right A-action on $D(H^*(RP^\infty))$ is the one naturally derived from the left A-action on $H^*(PR^\infty)$. The following is well known (see [5]).

Proposition 2.1 $Ext_A^{s,t}(Z_2, H^*(RP^\infty)) = Ext_A^{s,t}(D(H^*(RP^\infty)), Z_2)$ all s, t.

From this proposition we see to prove Theorem 1.2 amounts to proving that if Conjecture B is true then $Ext_A^{s,t}(D(H^*(RP^\infty)), Z_2) = 0$ for $t-s < 0$. We shall prove this by showing that $D(H^*(RP^\infty))$ is isomorphic to a quotient module of M as a left A-module where M is as in Conjecture B. Let M′ be the A-submodule of M generated by all x^i with $i > 0$. Let $M_1 = M/M'$. So M_1 is a vector space over Z_2 with base $\{\ldots, x^{-3}, x^{-2}\}$. The A-action on M_1 is given by $Sq^i x^{-k} = \binom{2^m - k}{i} x^{i-k}$ where m is large compared with i and k and when $i-k > -1$ we interpret $x^{i-k} = 0$ in M_1. Define a Z_2-isomrphism f: $M_1 \longrightarrow D(H^*(RP^\infty))$ by $f(x^{-k}) = y_{-k+1}$.

Proposition 2.2 f: $M_1 \longrightarrow D(H^*(RP^\infty))$ is a left A-module isomorphism of degree 1.

In order to prove Proposition 2.2 we need an identity in binomial coefficients (Lemma 2.3 (d) below). Given an integer $k \geq 1$ we write $_2{}^i$o.k. ε k if $_2{}^i$o.k. appears in the dyadic expansion of k, otherwise we write $_2{}^i \notin$ k. The proof of the following lemma is not hard (though very tedious) and is left to the reader.

Lemma 2.3 Let $k, i \geq 1$ be integers and let m be any integer which is large compared with i and k. Then

(a) $\binom{2^m-2^i-k-1}{2^i} = \begin{cases} 0 \pmod 2 & \text{if } 2^i \notin k \\ 1 \pmod 2 & \text{if } 2^i \varepsilon\, k, \end{cases}$

(b) $\binom{2^m-2^i-k}{2^i} = 0 \pmod 2$ in the following cases and $=1 \pmod 2$ otherwise

(i) k is odd and $2^i \notin k$

(ii) k is even, $2^i \notin k$ and $2^\ell \varepsilon\, k$ for some ℓ with $1 \leq \ell < i$

(iii) k is even, $2^i \varepsilon\, k$ and $2^\ell \notin k$ for any ℓ with $1 \leq \ell < i$,

(c) $\sum_{0<j<i} \binom{2^m-2^i+2^j-(k+1)}{2^i-2^j+1} = \begin{cases} 0 \pmod 2 \text{ if k is odd or if k is even and } 2^\ell \notin k \text{ for any } \ell \text{ with } 1 \leq \ell < 1 \\ 1 \pmod 2 \text{ if k is is even and } 2^\ell \varepsilon\, k \text{ for some } \ell \text{ with } 1 < \ell < i. \end{cases}$

So we have the following identity

(d) $\sum_{0 \leq j \leq i} \binom{2^m-2^i+2^j-(k+1)}{2^i-2^j+1} = \binom{2^m-2^i-k-1}{2^i} \pmod 2.$

Proof of Proposition 2.2: Let $D(M_1)$ be the right A-module which is related to M_1 in the way that $D(H^*(RP^\infty))$ is related to $H^*(RP^\infty)$. So $D(M_1)$ has a Z_2-base $\{z_2, z_3, \ldots\}$ with $\dim(z_k)=k$. Define a left A-action on $D(M_1)$ by $az_k = z_k \chi(a)$ where $\chi: A \longrightarrow A$ again is the canonical antiautomorphism of A. It is easy to see that the right A-action on $D(M_1)$ is given by $z_k\, sq^i = \binom{2^m-k-i}{i} z_{k+i}$ where m is large compared with i and k. So the left A-action on $D(M_1)$ is given by

$$\chi(Sq^i) z_k = \binom{2^m-k-i}{i} z_{k+i} \tag{1}$$

Define a Z_2-isomorphism $g: D(M_1) \longrightarrow H^*(RP^\infty)$ by $g(z_k) = u^{k-1}$. To prove the proposition is equivalent to proving that g is a left A-isomorphism of degree -1. In view of (1) we see this is equivalent to proving that in $H^*(RP^\infty)$

$$\chi(Sq^i)\, u^k = \binom{2^m-k-i-1}{i} u^{i+k}. \tag{2}$$

We prove (2) by induction on i and k. First, it is easy to see that (2) is true for i=1 and any k since $\chi(Sq^1)=Sq^1$. Next we prove that (2) is true for k=1 and any i. Recall that for any admissible sequence $I=(i_1, \ldots, i_{n+1})$ the action of Sq^I on u is given by

$$Sq^I u = \begin{cases} u^{2^{n+1}} & \text{if } I = (2^n, 2^{n-1}, \ldots, 2, 1) \\ 0 & \text{otherwise.} \end{cases}$$

It is well known ([4]) that $\chi(Sq^{2^{n+1}-1}) = Sq^{2^n} Sq^{2^{n-1}} \ldots Sq^2 Sq^1$. It follows that

$$\chi(Sq^i)u = \begin{cases} u^{2^{n+1}} & \text{if } i=2^{n+1}-1 \text{ for some n} \\ 0 & \text{otherwise.} \end{cases} \qquad (3)$$

On the other hand it is easy to check that

$$\binom{2^m-2-i}{i} = \begin{cases} 1 \pmod 2 & \text{if } i=2^{n+1}-1 \text{ for some n} \\ 0 \pmod 2 & \text{otherwise.} \end{cases}$$

This proves (2) for k=1 and any i. Now suppose $i > 1$ and $k > 1$ and suppose (2) is true for any pair (i′, k′) of positive integers such that either $i' < i$ or $k' < k$. Since $\{Sq^{2^j}\}_{j \geq 0}$ (and hence $\{\chi(Sq^{2^j})\}_{j > 0}$) is a Z_2-base for the indecomposable elements of the Steenrod algebra A, g: $D(M_1) \longrightarrow H^*(RP^\infty)$ is a Z_2-isomorphism and (1) is true for all i and k it follows that we may assume $i=2^\lambda$ for some λ. Let $\triangle: A \longrightarrow A \otimes A$ be the diagonal map of A. So $\triangle[\chi(Sq^{2^\lambda})] = \sum_{j=0}^{2^\lambda} \chi(Sq^{2^\lambda-j}) \otimes \chi(Sq^j)$. Then $\chi(Sq^{2^\lambda})u^k = \chi(Sq^{2^\lambda})(u^{k-1}.\, u)$

$$= \sum_{j=0}^{2^\lambda} \chi(Sq^{2^\lambda-j})u^{k-1}\chi(Sq^j)u \qquad \text{(Cartan formula)}$$

$$= \sum_{j=0}^{2^\lambda} \binom{2^m-2^\lambda+j-k}{2^\lambda-j} u^{2^\lambda-j+k-1}\chi(Sq^j)u \qquad \text{(by inductive hypothesis)}$$

$$= \sum_{0 \leq \nu \leq \lambda} \binom{2^m-2^\lambda+2^\nu-(k+1)}{2^\lambda-2^\nu+1} u^{2^\lambda+k} \qquad \text{[by (3)]}$$

$$=\binom{2^m-2\lambda-k-1}{2^\lambda} u^{2^\lambda+k}. \qquad \text{[by Lemma 2.3 (d)]}$$

This completes the inductive proof of (2). This proves proposition 2.2.

Now we prove Theorem 1.2.

Proof of Theorem 1.2: From Proposition 2.1 and Proposition 2.2 we see $E_2^{s,t}[E(RP^\infty), S^0]=Ext_A^{s,t}[Z_2, H^*(RP^\infty)]=Ext_A^{s,t-1}(M_1, Z_2)$. So to prove the theorem it suffices to show that $Ext_A^{s,t}(M_1, Z_2)=0$ for $t-s<-1$ provided that Conjecture B is true. The exact sequence of A-modules

$$0 \longrightarrow M' \xrightarrow{\ i\ } M \xrightarrow{\ j\ } M_1 \longrightarrow 0$$

yields the following long exact sequence in Ext groups

$$\cdots \longrightarrow Ext_A^{s-1,t}(M, Z_2) \xrightarrow{\ i^*\ } Ext_A^{s-1,t}(M', Z_2) \xrightarrow{\ \delta^*\ } Ext_A^{s,t}(M_1, Z_2) \xrightarrow{\ j^*\ } Ext_A^{s,t}(M, Z_2) \longrightarrow \cdots$$

Suppose Conjecture B is true. So $Ext_A^{s,t}(M, Z_2)=0$ all s, t. Then $Ext_A^{s-1,t}(M', Z_2)=Ext_A^{s,t}(M_1, Z_2)$ all s, t. Since M' begins with dimension zero we see $Ext_A^{s-1,t}(M', Z_2)=0$ if $t-s<-1$. Therefore $Ext_A^{s,t}(M_1, Z_2)=0$ if $t-s<-1$. This proves Theorem 1.2.

3. Proof of Theorem 1.3

Recall that Y_∞ is the homotopy inverse limit spectrum of the sequence $\cdots \longrightarrow Y_2 \longrightarrow Y_1 \longrightarrow Y_0 = S^0$. So we have the exact sequence

$$0 \longrightarrow \varprojlim_n{}^1 [X, Y_n]_{r+1} \longrightarrow [X, Y_\infty]_r \longrightarrow \varprojlim_n{}^0 [X, Y_n]_r \longrightarrow 0 \quad (4)$$

for any spectrum X and any integer r (See [1]).

Proposition 3.1 $\pi_i(Y_\infty)=0$ if $i < -1$, $\pi_{-1}(Y_\infty)=I_2/Z$, $\pi_0(Y_\infty)=0$, $\pi_i(Y_\infty)=\pi_i(S^0)/\pi_i(S^0;2)$ if $i \geq 1$ where I_2 = ring of 2-adic integers and $\pi_i(S^0;2)$ is the 2-componentt of the i^{th} stable homotopy group of spheres.

Proof: Since the mod 2 Adams resolution of S^0

$$\begin{array}{ccccc} & & \vdots & & \\ S^{-1}K_1 & \longrightarrow & Y_2 & \longrightarrow & K_2 \\ & & \downarrow & & \\ S^{-1}K_0 & \longrightarrow & Y_1 & \longrightarrow & K_1 \\ & & \downarrow & & \\ & & Y_0=S^0 & \longrightarrow & K_0 \end{array}$$

corresponds to the minimal resolution of Z_2 over the mod 2 Steenrod algebra A it follows from the already known behavior of $\mathrm{Ext}_A^{*,*}(Z_2, Z_2)$ that

(a) $\pi_i(Y_j)=0$ for $i \leq -1$ and and all j,

(b) $\pi_0(Y_j)=Z$ all j and each induced map $\pi_0(Y_j)=Z \longrightarrow \pi_0(Y_{j-i})=Z$ is the multiplication by 2 map,

(c) For each integer m, $m \geq 1$, $\pi_m(Y_k)=0$ for all large k.

Proposition 3.1 follows from (a), (b), (c) and the exact sequence (4).

Proof of Theorem 1.3: Let $K(G)$ be the Eilenberg-Maclane spectrum corresponding to an abelian group G. Let $H^*(RP^\infty; G)$ be the reduced cohomology of RP^∞ in the coefficient G. It is well known that if G is a vector space over the field Q of rationals or is a p-torsion group, p=odd prime, then $H^*(RP^\infty; G) = [(RP^\infty), K(G)]_{-*} = 0$.

By Proposition 3.1 $\pi_{-1}(Y_\infty)=I_2/Z$ and $\pi_i(Y_\infty)=0$ for $i < -1$. So there is a map $f: Y_\infty \longrightarrow S^{-1}K(I_2/Z)$ in S_h which induces an isomorphism $\pi_{-1}(Y_\infty)=I_2/Z \longrightarrow \pi_{-1}(S^{-1}K(I_2/Z))=I_2/Z$. Let W be the fiber of f. From Proposition 3.1 we see for each r $\pi_r(W)$ is an odd torsion group.

Apply $[E(RP^\infty), \text{——}]_*$ to the fiberation $W \longrightarrow Y_\infty \longrightarrow S^{-1}K(I_2/Z)$ to obtain the following long exact sequence

$$\cdots \longrightarrow [E(RP^\infty), W]_r \longrightarrow [E(RP^\infty), Y_\infty]_r \longrightarrow [E(RP^\infty), S^{-1}K(I_2/Z)]_r \longrightarrow [E(RP^\infty), W]_{r-1} \longrightarrow \cdots$$

Since $\pi_*(W)$ are odd torsion groups and $H^*(RP^\infty, Z_p) = 0$ for any odd prime p it follows from the obstruction theory that $[E(RP^\infty), W]_r = 0$ all r. Thus $[E(RP^\infty), Y_\infty]_r = [E(RP^\infty), S^{-1} K(I_2/Z)]_r = [E(RP^\infty), K(I_2/Z)]_{r+1}$. So to prove the theorem it suffices to prove that $[E(RP^\infty), K(I_2/Z)]_* = 0$.

Let $Z_{(2)}$ be the ring of integers localized at 2. Then $Z \subset Z_{(2)} \subset I_2$. The exact sequence $0 \longrightarrow Z_{(2)}/Z \longrightarrow I_2/Z \longrightarrow I_2/Z_{(2)} \longrightarrow 0$ induces a fiberation $K(Z_{(2)}/Z) \longrightarrow K(I_2/Z) \longrightarrow K(I_2/Z_{(2)})$ in S_h. It is well known that $I_2/Z_{(2)}$ is a vector space over Q; so

$[E(RP^\infty), K(I_2/Z_{(2)})]_*=0$ as remarked above. Therefore from the long exact sequence which is obtained by applying $[E(RP^\infty), \text{——}]_*$ to the fiberation $K(Z_{(2)}/Z) \longrightarrow K(I_2/Z) \longrightarrow K(I_2/Z_{(2)})$ we see $[E(RP^\infty), K(I_2/_2]_r =$ $[E(RP^\infty), K(Z_{(2)}/Z)]_r$ all r. It is also well known that

$Z_{(2)}/Z = \bigoplus_{p=\text{odd prime}} Z_{p^\infty}$ where Z_{p^∞} is the direct limit of the sepuence

$Z_p \longrightarrow Z_{p^2} \longrightarrow Z_{p^3} \longrightarrow \cdots$. So $K(Z_{(2)}/Z) = \times_{p=\text{odd prime}} K(Z_{p^\infty})$ in S_h. Since each Z_{p^∞} is a p-torsion group we see $[E(RP^\infty), K(Z_{p^\infty})]_r = 0$ all r provided p is an odd prime. Therefore $[E(RP_\infty), K(I_2/Z]]_r$

$[E(RP^\infty), K(Z_{(2)}/Z)]_r = \times_{p=\text{odd prime}} [E(RP^\infty), K(Z_{p^\infty})]_r = 0$ all r. This proves Theorem 1.3.

References

[1] J. F. Adams, Stable homotopy and generalized homology, Math. Lec. Notes. Univ. of Chicago, 1971.

[2] J. F. Adams, Operations of the n^{th} kind in K-theory and what we don't know about RP^∞, New Development in Topology, London Math. Lec. Note Series. 11, 1974.

[3] J. Boardman, Stable homotopy theory, mimeograph notes, Univ. of Warwick, Coventry, England, 1965.

[4] D. M. Davis, The antiautomorphism of the Steenrod algebra, Proc. Amer. Math. Soc. 44 (1974), 235-236.

[5] R. J. Milgram, The Steenrod algebra and its dual for connective K-theory, Nota De Matematicas Y Simposia, no. 1, 1975

Department of Mathematical Sciences
National Cheng-Chi University
Taipei, Taiwan, Rep. of China