A Note on Factorizations of Singular M-Matrices

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ABSTRACT

Supposing that M is a singular M-matrix, we show that there exists a permutation matrix P such that $PMP^T = LU$, where L is a lower triangular M-matrix and U is an upper triangular singular M-matrix. An example is given to illustrate that the above result is the best possible one.

I. INTRODUCTION

A real square matrix $A=(a_{i,j})$ is called an M-matrix if $a_{i,j} \leqslant 0$ whenever $i \neq j$ and all principal minors of A are positive. We will write $B=(b_{i,j}) \geqslant 0$ if $b_{i,j} \geqslant 0$ for each pair (i,j). For a real square matrix A with nonpositive off-diagonal elements, it is known (e.g., [1, Theorem 4.3]) that A is an M-matrix if and only if A is nonsingular and $A^{-1} \geqslant 0$. Following Fiedler and Ptak [1], we shall denote by K the class of all M-matrices and by K_0 the class of all real square matrices $A=(a_{i,j})$ with $a_{i,j} \leqslant 0$ for $i \neq j$, which have all principal minors nonnegative. A singular matrix in K_0 is called a singular M-matrix.

It is well known (e.g., [1, Theorem 4.3]) that an M-matrix may be written in the form LU, where $L \in K$ is lower triangular and $U \in K$ is upper triangular.

In [3], G. Poole and T. Boullion mentioned the possibility of the LU-factorizations for singular M-matrices. An example is given in Sec. 2 to show that not every matrix in K_0 can be factored as LU. However, for any matrix A in K_0, we show that $PAP^T = LU$ for a suitable permutation matrix P, where $L \in K$ is lower triangular and $U \in K_0$ is upper triangular.

The following result will be useful in our work.

THEOREM A [2, Theorem 4, p. 47]. *If a rectangular matrix R is represented in partitioned form*

$$R=\begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where A is a square nonsingular matrix of order n, then the rank of R is equal to n if and only if $D=CA^{-1}B$.

II. RESULTS

THEOREM 1. *Let* $M\in K_0$. *If M can be partitioned into the form*

$$M=\begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

such that A is nonsingular and $\operatorname{rank} M=\operatorname{rank} A$, *then* $M=LU$, *where* $L\in K$ *is lower triangular and* $U\in K_0$ *is upper triangular.*

Proof. We note first that $D=CA^{-1}B$ by Theorem A. Since $A\in K_0$ and A is nonsingular, we have $A\in K$. Thus, $A=L_1U_1$, where $L_1\in K$ is lower triangular and $U_1\in K$ is upper triangular. L_1 and U_1 are nonsingular; moreover, $L_1^{-1}\geqslant 0$ and $U_1^{-1}\geqslant 0$. Now let

$$L=\begin{bmatrix} L_1 & 0 \\ CU_1^{-1} & I \end{bmatrix} \quad\text{and}\quad U=\begin{bmatrix} U_1 & L_1^{-1}B \\ 0 & 0 \end{bmatrix},$$

where I is the identity matrix of appropriate order. Since $C\leqslant 0$ and $B\leqslant 0$, we have $CU_1^{-1}\leqslant 0$ and $L_1^{-1}B\leqslant 0$. Clearly, all principal minors of L are positive and all principal minors of U are nonnegative. Hence, $L\in K$ and $U\in K_0$, and $M=LU$. ■

COROLLARY. *Let* $M\in K_0$ *be irreducible. Then* $M=LU$, *where L and U are the same as in Theorem* 1.

Proof. If $M\in K$, then the statement is true. So we assume that M is singular. By Theorem 5.7 of [1], all proper principal minors of M are positive.

Thus, we can partition M into the form

$$M=\begin{bmatrix} M_{n-1} & b \\ c & d_{n,n} \end{bmatrix},$$

where $M_{n-1}\in K$ and $\operatorname{rank} M=\operatorname{rank} M_{n-1}$. Therefore, the corollary follows from Theorem 1. ■

Next, we prove a lemma.

LEMMA. *Let $M\in K_0$ be partitioned into the form*

$$M=\begin{bmatrix} A & B \\ 0 & D \end{bmatrix},$$

such that A and D are irreducible. Then $M=LU$, where L and U are the same as in Theorem 1.

Proof. It is clear that $A\in K_0$ and $D\in K_0$. By the above corollary $A=A_1A_2$ and $D=D_1D_2$, where A_1 and D_1 are lower traingular matrices in K, and A_2 and D_2 are upper triangular matrices in K_0. Let

$$L=\begin{bmatrix} A_1 & 0 \\ 0 & D_1 \end{bmatrix} \quad\text{and}\quad U=\begin{bmatrix} A_2 & A_1^{-1}B \\ 0 & D_2 \end{bmatrix}.$$

Then, $L\in K$ is lower triangular and $U\in K_0$ is upper triangular, and $M=LU$. ■

Our main result is the following.

THEOREM 2. *Let $M\in K_0$. Then there exists a permutation matrix P such that $PMP^T=LU$, where $L\in K$ is lower triangular and $U\in K_0$ is upper triangular.*

Proof. It is sufficient to consider the case that $M\neq 0$ is singular and reducible. Let P be a permutation matrix such that $\mathrm{PMP^T}$ can be partitioned into the form

$$PMP^T=\begin{bmatrix} A & B \\ 0 & D \end{bmatrix},$$

where A is irreducible. If D is also irreducible, then $PMP^T = LU$ by the Lemma. If D is reducible, then the proof is completed by using induction. ∎

It is clear that we can obtain another factorization for matrices in K_0, i.e., for any $M \neq 0$ in K_0, there exists a permutation matrix P such that $PMP^T = LU$, where $L \in K_0$ is lower triangular and $U \in K$ is upper triangular. Also, we can obtain similar results for factorizations of type UL.

EXAMPLE. The following example will show that Theorem 2 is the best possible result. Let

$$M = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

If

$$M = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix},$$

then we get $a_{11}b_{11} = 0$, $a_{11}b_{13} = -1$, and $a_{21}b_{11} = -1$, which is impossible. Thus, there is no factorization of the type LU for M. But if we let

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then

$$PMP^T = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = I \cdot \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

REFERENCES

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3 G. Poole and T. Boullion, A survey on M-matrices, *SIAM Review*, **16** (No. 4 1974), 419–427.