JOURNAL OF DIFFERENTIAL EQUATIONS 28, 35–42 (1978)

A Compact Positively Invariant Set of Solutions of the Nagumo Equation

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Received May 20, 1976; revised April 20, 1977

1. INTRODUCTION

The solutions of a parabolic partial differential equation can be considered as a semiflow in some function space. In order to apply the index theory (for example [5]) to study the qualitative behavior of the semiflow, it is often very useful to have a compact positively invariant set which is large enough to contain all intersting solutions (steady state solutions, traveling wave solutions, etc.).

The purpose of this paper is to find a compact positively invariant set of solutions of the Nagumo equation

$$\begin{aligned} u_t &= \alpha v - \beta u \\ v_t &= v_{xx} + f(v) - u, \end{aligned} \tag{1}$$

where u, v are real functions in $C^3(R \times R^+)$, $\alpha, \beta > 0$ and $f \in C^3(R)$ satisfying the following conditions:

(i) $f(0) = 0$

(ii) $f(-m) > (\alpha/\beta)m, f(m) < -(\alpha/\beta)m$ for large m.

We also assume that there exists $K > 0$ such that for every fixed t, $|u(x, t)|$ and $|v(x, t)|$ are less than e^{Kx^2} provided $|x|$ is sufficiently large.

A Compact Positively Invariant Set

For fixed t, we consider a solution $\binom{u}{v}$ of Eq. (1) as a curve $\Gamma^t: R \to R^4$ with parameter x, and

$$\Gamma^t(x) = \begin{pmatrix} u(x, t) \\ v(x, t) \\ u_x(x, t) \\ v_x(x, t) \end{pmatrix}.$$

* This paper forms part of the author's Ph.D. thesis written at the University of Wisconsin under the direction of Charles C. Conley. The author's research was supported in part by NSF MCS 76-07480.

0022-0396/78/0281-0035$02.00/0

We find a bounded closed region B in R^4 (see Fig. 2) such that if $\Gamma^0(x) \in B$ for all $x \in R$, then $\Gamma^t(x) \in B$ for all $(x, t) \in R \times R^+$.

2. A Bounded Positively Invariant Set

DEFINITION 1. A positively invariant set S means a subset of $C^3(R) \times C^3(R)$ such that every solution of Eq. (1) with initial value in S will stay in S for all $t \geqslant 0$.

DEFINITION 2. $S_1(m) = \{\binom{u}{v} \in C^3(R) \times C^3(R) \mid |v| \leqslant m \text{ and } |u| \leqslant (\alpha/\beta)m\}$. For fixed t, we consider a solution $\binom{u}{v}$ of Eq. (1) as a curve γ^t in R^2 with parameter $x \in R$ and $\gamma^t(x) = \binom{u(x,t)}{v(x,t)}$.

Roughly speaking, in the next lemma, we show that if γ^t is inside the rectangle in Fig. 1, and if it touches the boundary of the rectangle, it will be bounced inward. Therefore, once γ^t is in the rectangle, it will be trapped forever.

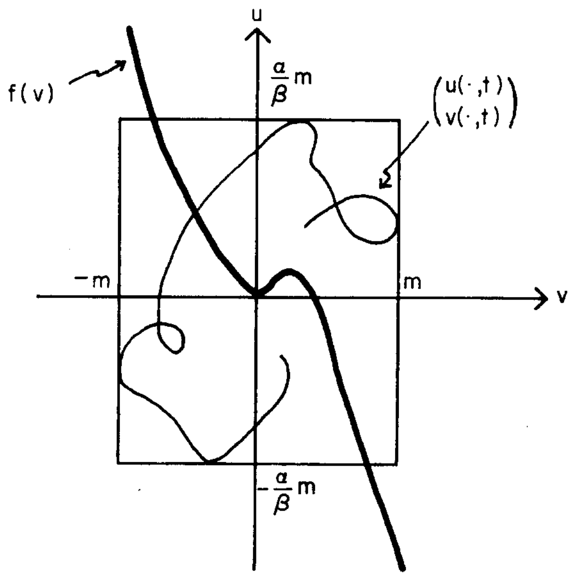

FIGURE 1

LEMMA 3. *For sufficiently large m, $S_1(m)$ is positively invariant.*

Proof. Let $\binom{u}{v}$ be a solution of Eq. (1) with the initial value $\binom{u(x,0)}{v(x,0)}$ in $S_1(m)$. Assume that for every fixed $t > 0$, $|u(x, t)|$, $|v(x, t)|$ are less than e^{Kx^2} for sufficiently large $|x|$.

Let $b = 4(K + 2)^2$, $a > 2(K + 2)$, $\epsilon > 0$ and $q(x, t) = e^{at+(bt+K+1)x^2}$.

Define two auxiliary functions:

$$\xi(x, t) = m + \epsilon q(x, t); \qquad \eta(x, t) = (\alpha/\beta)\xi(x, t).$$

It is easy to see that $|u(x, 0)| < \eta(x, 0)$ and $|v(x, 0)| < \xi(x, 0)$; and for every fixed t, if $|x|$ is sufficiently large, we have $|u(x, t)| < \eta(x, t)$ and $|v(x, t)| < \xi(x, t)$.

By simple computation, we have

$$q_t = (a + bx^2)q \quad \text{and}$$
$$q_{xx} = [4(bt + K + 1)^2x^2 + 2(bt + K + 1)]q.$$

Let $0 < t < 1/b$. Consider the following 4 cases:

(i) If $|u| \leqslant \eta$, $v = \xi$, $v_x = \xi_x$ and $v_{xx} \leqslant \xi_{xx}$, substituting v_t by (1), we have $\xi_t - v_t = \xi_t - (v_{xx} + f(v) - u) \geqslant \xi_t - \xi_{xx} - f(\xi) - \eta \geqslant [a + bx^2 - 4(bt + K + 1)^2x^2 - 2(bt + K + 1)]q - f(\xi) - \alpha/\beta > 0$, because $bt < 1$, $b = 4(K + 2)^2$, $a > 2(K + 2)$ and for sufficiently large ξ, $f(\xi) < -(\alpha/\beta)\xi$.

(ii) If $|u| \leqslant \eta$, $v = -\xi$, $v_x = -\xi_x$ and $v_{xx} \geqslant -\xi_{xx}$, then we have

$$\begin{aligned} -\xi_t - v_t &= -\xi_t - (v_{xx} + f(v) - u) \\ &\leqslant -\xi_t + \xi_{xx} - f(-\xi) + (\alpha/\beta)\xi < 0. \end{aligned}$$

(iii) If $|v| \leqslant \xi$, $u = \eta$, then we have

$$\begin{aligned} \eta_t - u_t &= \eta_t - (\alpha v - \beta u) \\ &\geqslant (\alpha/\beta)\,\epsilon(a + bx^2)q - \alpha\xi + \beta(\alpha/\beta)\xi > 0. \end{aligned}$$

(iv) If $|v| \leqslant \xi$, $u = -\eta$, then we have

$$-\eta_t - u_t \leqslant -(\alpha/\beta)q_t + \alpha\xi - \beta\eta < \alpha\xi - \beta(\alpha/\beta)\xi = 0.$$

Therefore, applying the Nagumo–Westphal lemma for systems of parabolic equations (see [4]), we have $|v(x, t)| < \xi(x, t)$ and $|u(x, t)| < \eta(x, t)$ for $0 \leqslant t < 1/b$.

Since ϵ can be arbitrarily small we have $|u(x, t)| \leqslant (\alpha/\beta)m$ and

$$|v(x, t)| \leqslant m \qquad \text{for} \quad 0 \leqslant t < 1/b.$$

The above argument can be applied again and again, hence

$$|u(x, t)| \leqslant (\alpha/\beta)m \qquad \text{and} \qquad |v(x, t)| \leqslant m \qquad \text{for all} \quad t \geqslant 0.$$

3. A Compact Positively Invariant Set

$S_1(m)$ is bounded but not compact. Now we find a positively invariant subset of $S_1(m)$ whose elements have uniformly bounded derivatives.

At first we have to develop some technical results.

Consider the autonomous system of ordinary differential equations.

$$
\begin{aligned}
dv/dx &= w \\
dw/dx &= -f(v) \mp vw.
\end{aligned} \tag{$2^\pm$}
$$

In the v–w phase space, let $\tilde{w}^+(v)$ ($\tilde{w}^-(v)$) be the orbit of the system (2^+) (system (2^-)) passing through the point $\tilde{w}^+(0) = (\beta/2\alpha)n$ ($\tilde{w}^-(0) = -(\beta/2\alpha)n$, respectively), where

$$m_1 = \max_{u,v\in S_1(m)} \{|\, u\,|, |\, v\,|, |f(v)|, 1\} \tag{3}$$

$$n = \max\{8m_1^2(\alpha/\beta), 2m_1^2, 32(\alpha^2/\beta^2)\}. \tag{4}$$

Note. The definitions of $\tilde{w}^\pm$, and m_1, n will be used throughout this paper.

LEMMA 4. *$1 < |\, \tilde{w}^\pm(v)| < (\beta/\alpha)n$ for $|\, v\,| \leqslant m_1$.*

Proof. From Eq. ($2^\pm$), it follows

$$\frac{d\tilde{w}^\pm}{dv}\tilde{w}^\pm = -f(v) \mp v\tilde{w}^\pm.$$

Assuming that $|\, \tilde{w}^\pm \,| \geqslant 1$, and using the above formula and (3), we get

$$\left|\frac{d\tilde{w}^\pm}{dv}\right| \leqslant \left|\frac{f(v)}{\tilde{w}^\pm}\right| + |\, v\,| \leqslant 2m_1. \tag{5}$$

But $\tilde{w}^\pm(v)$ starts at $|\, \tilde{w}^\pm(0)| = \frac{1}{2}(\beta/\alpha)n > 1$, therefore by (5), if $|\, v\,| \leqslant m_1$, we have

$$|\, \tilde{w}^\pm(v)| \geqslant \tilde{w}^\pm(0) - \left|\frac{d\tilde{w}^\pm}{dv}\right| |\, v\,|$$

using (3), (4), (5) (6)

$$
\begin{aligned}
&\geqslant \tfrac{1}{2}(\beta/\alpha)\, n - 2m_1^2 \\
&\geqslant \tfrac{1}{4}(\beta/\alpha)\, n > 1.
\end{aligned}
$$

So the above assumption is a fact.

Using (5) again and recalling (3), (4), we have

$$
\begin{aligned}
|\, \tilde{w}^\pm(v)| &\leqslant \tilde{w}^\pm(0) + \left|\frac{d\tilde{w}}{dv}\right| |\, v\,| \\
&\leqslant \tfrac{1}{2}(\beta/\alpha)\, n + 2m_1^2 \\
&\leqslant \tfrac{1}{2}(\beta/\alpha)\, n + \tfrac{1}{4}(\beta/\alpha)\, n < (\beta/\alpha)\, n.
\end{aligned} \tag{7}
$$

Thus we have proved the lemma. ∎

DEFINITION 5. $S_2(m) = \{\binom{u}{v} \in S_1(m) \mid |u_x| < n \text{ and } \tilde{w}^-(v(x)) \leqslant v_x(x) \leqslant \tilde{w}^+(v(x)) \text{ for } x \in R\}$ (see Fig. 2).

THEOREM 6. *$S_2(m)$ is positively invariant for large m.*

Before we prove the theorem, we need to establish some identities.

Let $\binom{u}{v}$ be a solution of Eq. (1). Set $w = v_x$, $z = u_x$. Differentiating Eq. (1) wrt x, we get

$$z_t = \alpha w - \beta z \tag{8}$$

$$w_t = w_{xx} + f'(v)w - z \tag{9}$$

When $v_x \neq 0$, locally we can consider z, w as functions of v and t. Define

$$\bar{z}(v, t) = \bar{z}(v(x, t), t) = z(x, t), \tag{10}$$

$$\bar{w}(v, t) = \bar{w}(v(x, t), t) = w(x, t). \tag{11}$$

Through a change of variable, Eqs. (8)–(9) can be rewritten as:

$$\begin{aligned} \bar{z}_t &= -\bar{z}_v v_t + z_t \\ &= -\bar{z}_v(\bar{w}_v w + f(v) - u) + \alpha\bar{w} - \beta\bar{z} \end{aligned} \tag{12}$$

and

$$\begin{aligned} \bar{w}_t &= -\bar{w}_v v_t + w_t \\ &= \bar{w}^2\bar{w}_{vv} - f(v)\bar{w}_v + f'(v)\bar{w} - \bar{z} + \bar{w}_v u \end{aligned} \tag{13}$$

These will be needed later.

Now let us consider the ordinary differential equation ($2^\pm$) which can be written as:

$$d^2v/dx^2 + f(v) = \mp vw. \tag{14$^\pm$}$$

When $dv/dx \neq 0$, locally we can define:

$$\hat{w}(v) = \hat{w}(v(x)) = w(x) \tag{15}$$

Using ($14^\pm$), we get the following identity

$$-\frac{d\hat{w}}{dv}\left(\frac{d^2v}{dx^2} + f(v)\right) + \frac{d}{dx}\left(\frac{d^2v}{dx^2} + f(v)\right) = -\frac{d\hat{w}}{dv}(\mp vw) + \frac{d}{dx}(\mp vw),$$

which can be simplified as:

$$\hat{w}^2\frac{d^2\hat{w}}{dv^2} + f'(v)\,\hat{w} - f(v)\,\hat{w}_v = \mp\hat{w}^2. \tag{16$^\pm$}$$

Proof of Theorem 6. Let $\binom{u}{v}$ be a solution of Eq. (1), $w = v_x$, $z = u_x$, $\bar{z}$, $\bar{w}$ be defined as in (10), (11). For fixed t, we consider $\binom{u}{v}$ as a curve Γ^t in R^4 by defining

$$\Gamma^t(x) = \begin{pmatrix} u(x, t) \\ v(x, t) \\ u_x(x, t) \\ v_x(x, t) \end{pmatrix}.$$

Let B be the closed region in R^4 bounded by $v = \pm m$, $u = \pm(\alpha/\beta)m$, $z = \pm n$, $w = \tilde{w}^+(v)$, and $w = \tilde{w}^-(v)$ (see Fig. 2). We show that if Γ^t is in B and if it touches the boundary of B, it will be bounced inward.

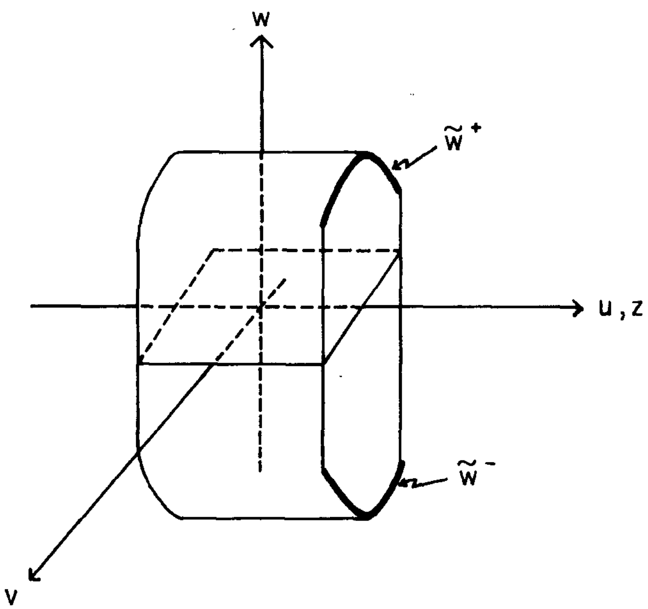

FIGURE 2

In Lemma 3 we have covered the cases of Γ^t touching the boundary in u, v directions, so we only need to discuss the following two cases.

Case 1. If Γ^t touches the z directional boundary at $x = x_0$, then $z(x_0, t) = n$ and $z_x(x_0, t) = 0$. Applying (8) and Lemma 4 we have

$$z_t(x_0, t) = \alpha w - \beta z \leqslant \alpha \mid \tilde{w}^{\pm} \mid - \beta z$$
$$< \alpha \cdot (\beta/\alpha)n - \beta n \leqslant 0.$$

Case 2. If Γ^t touches the w directional boundary at $v = v_0$, then we have

$$\bar{w}(v_0, t) = \tilde{w}^+(v_0),$$
$$\bar{w}_v(v_0, t) = \frac{d\tilde{w}^+}{dv}(v_0)$$

and (17)

$$\bar{w}_{vv}(v_0, t) \leqslant \frac{d^2\tilde{w}^+}{dv^2}(v_0).$$

Using (13), at the point (v_0, t), we have

$$\begin{aligned}\bar{w}_t &= \bar{w}^2\bar{w}_{vv} - f(v_0)\,\bar{w}_v + f'(v_0)\,\bar{w} - \bar{z} + \bar{w}_v u \\ &\leqslant (\tilde{w}^+)^2 \frac{d^2\tilde{w}^+}{dv^2} - f(v_0)\,\tilde{w}_v^+ + f'(v_0)\,\tilde{w}^+ - \bar{z} + \tilde{w}_v^+ u \quad \text{by (17)} \\ &= -(\tilde{w}^+)^2 - \bar{z} + \tilde{w}_v^+ u \quad \text{by (16)}\end{aligned}$$

using the definition of B and (6), (5), and (3)

$$\begin{aligned}&\leqslant -(\tfrac{1}{4}(\beta/\alpha)\,n)^2 + n + 2m_1^2 \\ &= -\tfrac{1}{16}(\beta^2/\alpha^2)\,n \cdot n + n + 2m_1^2 \\ &\leqslant -2n + n + 2m_1^2 \quad \text{by (4)} \\ &\leqslant -n + 2m_1^2 < 0 \quad \text{by (4)}.\end{aligned}$$

The same argument can be applied to the cases Γ^t touches $z = -n$ or $w = \tilde{w}^-(v)$.

Therefore, once $\binom{u}{v}$ is in $S_2(m)$, it will stay in $S_2(m)$ forever. ▮

THEOREM 7. *$S_2(m)$ is compact* wrt *c–O topology.* (*i.e., the topology generated by sup-norms of u, v restricted to every bounded interval of the real line.*)

Proof. Let $\binom{u}{v} \in S_2(m) \cdot u, v, u_x, v_x$ are uniformly bounded. Hence by Ascoli–Azela Theorem, $S_2(m)$ is compact wrt *c–O* topology.

ACKNOWLEDGMENT

The author would like to thank Professor Charles C. Conley for his helpful advice.

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