Chapter 3

Precautionary Saving and Consumption with Borrowing Constraints

3.1 Introduction

The permanent income hypothesis (PIH) roots in the consumer's optimal consumption decision which is to maximize his expected life-time utility. While being conceptually accepted by most macroeconomic researchers, there have been no lack of empirical rejections of the model accumulated in the literature. According to the PIH, the optimal consumption level chosen by consumer is determined by his permanent income.[1] Changes in transitory income should not affect this consumption level. Yet when investigating the consumption data, it is often found that the consumption is sensitive to the current income level, which violates what the PIH predicts (Flavin, 1981, 1985; Zeldes, 1989a, Attanasio and Weber, 1995; Shea, 1995; Lusardi, 1996). Untangling what causes this excess sensitivity of consumption to current income then constitutes the main goal of many researches.[2]

Among the many reasons discussed in the literature, the presence of the borrowing constraints is often found to be responsible for the empirical rejections of the PIH (Zeldes, 1989a; Japppelli and Pagano, 1989; Bacchetta and Gerlach, 1997; Jappelli, et al., 1998;

[1]The permanent income is defined as consumer's expected present value of his life-time wealth.

[2]The excess sensitivity of consumption is defined as the phenomenon that the consumption level chosen by the consumer is sensitive to current income, such as Flavin (1987); or the situation where the marginal propensity of consumption to current income is greater than what the PIH implies, such as Hall and Mishkin (1982).

Ludvigson, 1999). This imperfection in the credit market often results from the information asymmetry problem between the creditors and the debtors. When potential credit suppliers cannot directly observe the creditworthiness of the loan demanders; or when they cannot directly monitor the actions taken by their debtors after the loan has been transferred, their incentives to participate in the credit market would be very low, or they simply impose additional requirements for the loans they supply. If the consumers cannot acquire enough loans to finance their optimal consumption plan, their consumption level would be confined to the current resource available. The consumption exhibits excess sensitivity as a result.

While often being attributed to as the direct cause of the empirical failure of the PIH, theoretical justifications on the consumption behavior with the presence of the borrowing constraints are found to be too few. Related researches (such as Scheinkman and Weiss, 1986; Deaton, 1991; and Carroll, 2001) often resort to the numerical analysis as the closed-form solutions are not attainable in the general setting when there are borrowing constraints facing consumers. Although the numerical solution technique provides a practical alternative way for the analysis, it is often subject to the critique that the economic decisions are made within the 'black box' when numerical solutions are used. Dow and Olsen (1991) thus construct a three-period stochastic environment and provide the analytic solution to the model, where consumers face quadratic utility function and the possibly binding borrowing constraints. Following their work, we discuss in detail the consumption and saving behavior with the analytical solution derived in this chapter. The empirical implications of the theory are also drawn.

We first discuss the precautionary saving behavior induced by the borrowing constraints. It is well known that the quadratic utility function processes the certainty equivalence property, were there no borrowing constraints in the model. We therefore can isolate the precautionary saving behavior that purely results from the presence of the borrowing constraints if consumers now respond to uncertainties facing them. The precautionary saving is defined as savings consumers engage in to buffer future uncertainties. Conditions to induce precautionary behavior include the convexity in the marginal utility and the presence of the future income uncertainties (Zeldes, 1989b; Kimball, 1990). The random walk hypothesis implied by the traditional PIH is derived in the environment where the utility exhibits certainty equivalence, or where consumers can perfectly foresee their future income. It thus cannot account for the precautionary saving incentives of consumers.

The reason why introducing the borrowing constraints might induce precautionary saving lies on consumer's incentive to smooth his consumption. When borrowing constraints are absent, the consumer can buffer any external shock through unlimited borrowing and lending to maintain a smooth consumption pattern. That is, through borrowing and lending in the credit market, the effects of the shock can then be spreaded evenly into the remaining of his life. The smoothness of his consumption are thus restored. When there are borrowing constraints facing the consumer, however, his ability to buffer shocks through the credit market is limited. External shocks may lead to great volatilities in consumption, which violates his optimal consumption plan. He then 'saves for the raining days' when he has the ability to save to self-finance his optimal consumption plan. The amount of savings accumulated by the consumer is then directly linked to the extent of uncertainties he perceive. The precautionary saving behavior is induced as a result. Later on, we will provide theoretical underpinning of this precautionary saving behavior as the marginal value function now exhibit convexity when the borrowing constraints are present.

We also show that the precautionary saving behavior leads to a concave consumption function. This consumption pattern is similar to that when borrowing constraints are absent and the utility function specification that has positive third-derivatives, such as the CRRA utility function (Zeldes, 1989b; Kimball, 1990; Carroll and Kimball, 1996). It is also consistent with the empirical finding of Lusardi (1996), Souleles (1999) and Parker (1999) that the marginal propensity to consume of the poor is greater than that of the rich.

Comparative statics are also conducted, since we can now derive the analytical solution of the model. The consequences of changes in consumer's expected income, income volatility, as well as the possibility of being constrained in the future are all discussed. The consumption now responds to changes in various moments of the income process. Taking the effects of change in the probability of being borrowing constrained as an example, the effects are all channeled through the influences on the expected future income, income volatility, as well as the higher-order moments such as the skewness of the income distribution. Our analysis also sheds light on empirical studies of the consumption behavior. These empirical implications include the cause of the excess sensitivity, the consequences of the sample splitting between the rich and the poor, as well as the relevance of the higher-order moments to the consumption dynamics.

The rest of this chapter is organized as follows. Section 2 describes the model setup and Section 3 provides the analytical solution to the model. Implications of the solution are discussed in Section 4; comparative static analysis are provided in Section 5. In Section 6, we discuss the empirical implications of the model, and Section 7 then concludes.

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3.2 Model Setup

Consumers maximize their expected discounted lifetime utility,

$$\underset{\{C_s\}_{s=t}^{T}}{\text{Max}} \quad E_t\left[\sum_{s=t}^{T}\beta^{s-t}U(C_s)\right], \tag{3.1}$$

where E_t represents the expectation conditional on all information available at time t; C_s equals the real consumption at period s; $0 < \beta \leq 1$ is consumers' subjective discount factor; and $U(\cdot)$ is the one-period, time-separable utility function. We assume that the utility function is a linear-quadratic one, with marginal utility $U'(C) = \alpha - \theta C$, where $\alpha \geq 0$, $\theta > 0$, and $U'(C) > 0, U''(C) < 0$. This utility specification has its advantage in that the optimal decision rules can be solved analytically. Moreover, as this utility exhibits certainty equivalence property, the precautionary saving induced by the presence of the borrowing constraints can thus be isolated. We further assume that the only uncertainty underlying this economy is the future labor income Y, which is either Y_H, high income, or Y_L, low income, $Y_H > Y_L$, with probability p and $1-p$, respectively. Expected future income is denoted by $\bar{Y}$ $(= pY_H + (1-p)Y_L)$.

Consumers face the following dynamic budget constraint,

$$B_{s+1} - B_s = rB_s + Y_s - C_s, \tag{3.2}$$

where B_s represents the risk-free bond accumulated form period $s-1$. Optimal consumption plan can be achieved through borrowing and lending in the bond market. The bond pays a fixed real interest rate r at each period, and the consumption/saving decisions made by consumers do not affect this real interest rate. Equation (3.2) thus reveals consumer's allocation of their labor income and interest income/payment into consumption and the accumulation of wealth. By defining $W_s = (1+r)B_s + Y_s$ as the initial wealth holding of the consumer at period s, the budget constraint can be rewritten as the following transition equation:

$$\begin{aligned} W_{s+1} &= (1+r)B_{s+1} + Y_{s+1} \\ &= (1+r)[(1+r)B_s + Y_s - C_s] + Y_{s+1} \\ &= (1+r)[W_s - C_s] + Y_{s+1}. \end{aligned} \tag{3.3}$$

Most consumers cannot borrow without a limit. As there are information asymmetry problems in the capital market (Stiglitz and Weiss, 1981; Mankiw, 1986), consumers may have limited access to the capital market and thus been restrained from their optimal consumption plan. To capture this economic reality, we assume that consumers face the following borrowing constraints:

$$B_s \geq -b; s = t, t+1, \ldots, T-1, \tag{3.4}$$

were $b \geq 0$ is the maximum amount individuals can borrow.

3.3 Optimal Consumption with Borrowing Constraints

In this section, we follow Dow and Olsen (1991) with the three-period, quadratic utility specification, and the solution concepts provided therein. The optimal consumption functions are to be derived analytically, and the precautionary saving induced by the presence of the borrowing constraints are to be calculated explicitly.

Consumers now maximize their expected lifetime utility (3.1), subject to the constraints (3.2) and (3.4). For the computational simplicity, we assume that $b = 0$. Consumers are prohibited from borrowing in the bond market, but they maintain the unlimited access to the saving behavior. With these borrowing constraints, consumers may only consume up to their beginning-of-period wealth, which equals the sum of their current income and the savings accumulated from previous periods.[3] We now solve the model recursively from the last period, with the typical dynamic programming technique.

The consumer consumes all resources available at the terminal period of his life. Thus $C_3 = W_3$, $V_3(W_3) = U(W_3)$, and $V_3'(W_3) = U'(W_3)$. The optimizing problem facing the consumer at period 2 is then:[4]

$$\begin{aligned} V_2(W_2) &= \underset{C_2 \leq W_2}{\text{Max}} \; U(C_2) + \beta E_2 V_3(W_3) + \lambda_2(W_2 - C_2) \\ &= \underset{C_2 \leq W_2}{\text{Max}} \; U(C_2) + \beta E_2 V_3[(1+r)(W_2 - C_2)] + \lambda_2(W_2 - C_2), \end{aligned}$$

where λ_2 is the Lagrangian multiplier of period-2 borrowing constraint. First-order condition of this optimizing problem is therefore

$$\begin{aligned} &U'(C_2) - E_2 V_3'(W_3) - \lambda_2 = 0; \\ &\lambda_2(W_2 - C_2) = 0; \\ &\lambda_2 \geq 0, W_2 \geq C_2. \end{aligned}$$

The period-2 consumption function can then be derived as:[5]

$$C_2 = \begin{cases} \frac{1}{2+r}[(1+r)W_2 + \bar{Y}] & \text{if } W_2 \geq \bar{Y}; \\ W_2 & \text{if } W_2 < \bar{Y}. \end{cases} \tag{3.5}$$

[3] This assumption seems to be a restrictive one, but analytically, it is essentially equivalent to setting the borrowing limit to an arbitrary constant.

[4] The budget constraint (3.2) implies $B_{s+1} = [(1+r)B_s + Y_s] - C_s = W_s - C_s$. Thus setting $B_s \geq 0$ is equivalent to restricting $C_s \leq W_s$.

[5] $C_2 - W_2 = \frac{1}{2+r}[(1+r)W_2 + \bar{Y}] - W_2 = \frac{\bar{Y} - W_2}{2+r}$, $C_2 \leq W_2$ if $W_2 \geq \bar{Y}$. Thus C_2 is not constrained if $W_2 \geq \bar{Y}$.

Whether the period-2 consumption is constrained depends on consumers' wealth holding at that period. Since W_2 is the sum of the current income Y_2 and the value of assets accumulated from previous periods, the optimal consumption will not be constrained if the current income is high, or if consumers have accumulated enough assets from previous periods. Consequently, according to the envelope theorem, the marginal value function of period 2 is

$$V_2'(W_2) = \begin{cases} \alpha - \theta\{\frac{1}{2+r}[(1+r)W_2 + \bar{Y}]\} & \text{if } W_2 \geq \bar{Y}; \\ \alpha - \theta W_2 & \text{if } W_2 < \bar{Y}. \end{cases} \tag{3.6}$$

At period 1, the consumer solves the following optimizing problem:

$$\begin{aligned} V_1(W_1) &= \underset{C_1 \leq W_1}{\text{Max}} \; U(C_1) + \beta E_1 V_2(W_2) + \lambda_1(W_1 - C_1) \\ &= \underset{C_1 \leq W_1}{\text{Max}} \; U(C_1) + \beta E_1 V_2[(1+r)(W_1 - C_1)] + \lambda_1(W_1 - C_1), \end{aligned}$$

where λ_1 is the Lagrangian multiplier corresponding to the period-1 borrowing constraint. This yields the following first-order conditions:

$$\begin{aligned} &U'(C_1) - E_1 V_2'(W_2) - \lambda_1 = 0; \\ &\lambda_1(W_1 - C_1) = 0; \\ &\lambda_1 \geq 0, W_1 \geq C_1. \end{aligned} \tag{3.7}$$

Before deriving the optimal period-1 consumption function, we introduce first the two 'thresholds' of period-1 wealth that are relevant for our following discussion. It is now evident from the previous analysis that for a given Y_2, whether the period-2 consumption is constrained depends solely on how much assets the consumer has accumulated from the previous periods. W_1 thus determines not only whether the optimal C_1 is constrained, but also whether the optimal C_2 will be borrowing constrained. We can therefore calculate some initial wealth level, say, W_H, above which the consumer's optimal consumption is not constrained at both period one and period two, whatever the realization of Y_2 is. That is, if the consumer's initial wealth is high enough (greater than W_H), both C_1 and C_2 are not unconstrained even if the consumer receives low income in period two.

Table 3.1: Thresholds and the Borrowing Constraints

W_1	C_1	C_2
$W_1 > W_H$	unconstrained	unconstrained
$W_H \geq W_1 \geq W_L$	unconstrained	constrained if $Y_2 = Y_L$; unconstrained if $Y_2 = Y_H$
$W_1 < W_L$	constrained	constrained if $Y_2 = Y_L$; unconstrained if $Y_2 = Y_H$

To the other extreme, if W_1 is relatively low (less than some given wealth level W_L), the optimal period-1 consumption will exceed W_1. C_1 is therefore constrained and no asset will be left to the following periods. In this case, C_2 will be unconstrained if $Y_2 = Y_H$; constrained if $Y_2 = Y_L$. In between, if W_1 lies between these two thresholds, C_1 is not constrained but the asset left for period 2 is not sufficient to make C_2 unconstrained regardless of the actual income realization. These results are summarized in Table3.1.

We proceed first with the derivation of W_H and W_L. As shown in Table3.1, if $W_1 > W_H$, both C_1 and C_2 are unconstrained. W_H can be viewed as the 'minimum' wealth level above which consumers can consume their optimal consumption level without being constrained. This is the vary marginal wealth level that makes period 2 consumption just constrained if the low income is realized at period two. From equation (3.5), the condition is held only if $(1+r)B_2 + Y_L = W_2 \geq \bar{Y}$. Substituting $B_2 = W_1 - C_1$ into it yields

$$C_1 \leq W_1 - \frac{(\bar{Y} - Y_L)}{1+r}.$$

Thus C_1 is unconstrained. By substituting the optimal consumption of the unconstrained economy into the proceeding equation, the minimum wealth level that makes both C_1 and C_2 unconstrained is then:

$$\begin{aligned} W_1 &\geq \bar{Y} + \frac{1}{1-k_3}(\bar{Y} - Y_L) \\ &\equiv W_H. \end{aligned} \tag{3.8}$$

When contrasting with W_H, W_L can be regarded as the 'maximum' wealth level, below which period-one consumption is borrowing constrained. We know that if C_1 is constrained, $C_1 = W_1$, $B_2 = 0$, and $W_2 = Y_2$. Whether C_2 is constrained thus depends solely on the income level realized at that period. That is, C_2 will be constrained if $Y_2 = Y_L$; unconstrained if $Y_2 = Y_H$. Equation (3.6) then yields

$$V'(W_2) = \begin{cases} \alpha - \theta\left\{\frac{1}{2+r}[(1+r)Y_H + \bar{Y}]\right\} & \text{if } Y_2 = Y_H; \\ \alpha - \theta Y_L & \text{if } Y_2 = Y_L. \end{cases}$$

The corresponding first-order condition is

$$\begin{aligned} U'(W_1) &= E_1 V_2'(W_2) + \lambda_1 \Leftrightarrow \\ \alpha - \theta W_1 &= \alpha - \theta \left[\left(\frac{1+r+p}{2+r} \right) \bar{Y} + \left(\frac{1-p}{2+r} \right) Y_L \right] + \lambda_1, \end{aligned}$$

where $\lambda_1 \geq 0$ and $\theta \geq 0$. This implies

$$W_1 - \left[\left(\frac{1+r+p}{2+r} \right) \bar{Y} + \left(\frac{1-p}{2+r} \right) Y_L \right] = \frac{-\lambda_1}{\theta} \leq 0.$$

Given r, p, θ, $\bar{Y}$, Y_L, we can solve for W_1 that makes $\lambda_1 > 0$ (C_1 constrained). W_L is then the upper bound of these wealth levels

$$\begin{aligned} W_1 &\leq \left(\frac{1+r+p}{2+r} \right) \bar{Y} + \left(\frac{1-p}{2+r} \right) Y_L \\ &= \bar{Y} - \frac{1-p}{2+r}(\bar{Y} - Y_L) \\ &\equiv W_L. \end{aligned} \tag{3.9}$$

As discussed previously, W_1 determines whether the current and future consumption will be borrowing constrained. We therefore have to derive the distinct consumption rules when W_1 lies in different regions constructed by the two thresholds.

3.3.1 High Wealth Level ($W_1 > W_H$)

The wealth level is sufficient to ensure both the current and the future consumption against being constrained. The consumer behaves as if there were no borrowing constraints in the economy. The optimal consumption is thus

$$C_1^B = \bar{Y} + \frac{(1+r)^2}{1+(1+r)+(1+r)^2}(W_1 - \bar{Y}). \tag{3.10}$$

3.3.2 Middle Wealth Level ($W_L \leq W_1 \leq W_H$)

The period-2 marginal value function is

$$V'(W_2) = \begin{cases} \alpha - \theta\{\frac{1}{2+r}\bar{Y} + \frac{1+r}{2+r}[(1+r)(W_1 - C_1) + Y_H]\} & \text{if } Y_2 = Y_H; \\ \alpha - \theta[(1+r)(W_1 - C_1) + Y_L] & \text{if } Y_2 = Y_L. \end{cases}$$

Since the optimal C_1 is not constrained within this wealth region, the period-one first-order condition would be

$$U'(C_1) = E_1 V_2'(W_2).$$

Given that the consumer will be constrained with probability $1-p$ (low income); unconstrained with probability p (high income) in period two, this first-order condition can be written as

$$C_1 = p\left\{\frac{1}{2+r}\bar{Y} + \frac{1+r}{2+r}\left[(1+r)(W_1 - C_1) + Y_H\right]\right\} + (1-p)[(1+r)(W_1 - C_1) + Y_L].$$

Period-1 optimal consumption is thus

$$\begin{aligned} C_1^B &= \frac{1}{(2+r)^2 - p(1+r)}[(1+r+p)\bar{Y} + (1+r)(2+r-p)W_1 + (1-p)Y_L] \\ &= \bar{Y} + \frac{(1+r)(2+r-p)}{(2+r)^2 - p(1+r)}(W_1 - \bar{Y}) - \frac{1-p}{(2+r)^2 - p(1+r)}(\bar{Y} - Y_L). \end{aligned} \tag{3.11}$$

The optimal consumption level is now different from the unconstrained counterpart, although the consumer is capable of consuming as much as the unconstrained case. This reveals the fact that the *possibility* that the consumer may be constrained in the future does alter the optimal consumption level chosen by him. In addition, as long as W_1 lies between the two thresholds, the optimal level chosen by the consumer would never be greater than the unconstrained case.[6] The consumption behavior is now more conservative than the unconstrained case. What's more, an additional term $(\bar{Y} - Y_L)$ now enters into the optimal consumption function. The consumer cares not only the present value of his expected lifetime wealth, but also to which extent Y_L deviates from his expected income.

[6] Optimal consumption level without the presence of borrowing constraints is $C_1^N = \bar{Y} + \frac{(1+r)^2}{1+(1+r)+(1+r)^2}(W_1 - \bar{Y})$.

The reason why $(Y_H - \bar{Y})$ does not affect the consumption level stems form our asymmetric setting that only borrowing is constrained. Since we do not restrict the lending behavior, the effect of temporarily high income on consumption can be smoothed out by saving now and receiving the proceeds in subsequent periods when their income is not that high. In contrast, efforts to smooth low income may be restrained by borrowing constraints. The consumer thus has to consider how low his income may be in making consumption decision. Moreover, we can see from equation (3.11) that C_1^B declines with the increase of $(\bar{Y} - Y_L)$. We will return to these findings and their economic intuitions later in the next section.[7]

3.3.3 Low Wealth Level $(W_1 < W_L)$

C_1^N is not attainable within this wealth region. The marginal utility to consume now is greater than that of the subsequent periods, Consumers consume all resources available at this period. The consumption function is thus

$$C_1^B = W_1. \tag{3.12}$$

Consumers do not engage in the intertemporal substitution, because the marginal utility is very high for the relatively low period-one consumption.[8]

[7]Period-2 and period-3 optimal consumption levels can also be calculated after C_1^B is derived. According to the realized income level, the optimal consumption of period two is

$$C_2^B = \begin{cases} \bar{Y} + \frac{(1+r)^2}{(2+r)^2 - p(1+r)}(W_1 - \bar{Y}) + \frac{(1-p)(1+r)^2}{(2+r)[(2+r)^2 - p(1+r)]}(\bar{Y} - Y_L) + \frac{1+r}{2+r}(Y_H - \bar{Y}) & \text{if } Y_2 = Y_H; \\ \bar{Y} + \frac{(1+r)(2+r)}{2(1+r)^2 - p(1+r)}(W_1 - \bar{Y}) - \frac{r^2+3r+3}{2(1+r)^2 - p(1+r)}(\bar{Y} - Y_L) & \text{if } Y_2 = Y_L. \end{cases}$$

Period-3 consumption depends on its realized income and wealth accumulated form the previous periods.

$$C_3^B = \begin{cases} Y_3 + \frac{(1+r)^2}{(2+r)^2 - p(1+r)}(W_1 - \bar{Y}) + \frac{(1-p)(1+r)^2}{(2+r)[(2+r)^2 - p(1+r)]}(\bar{Y} - Y_L) + \frac{1+r}{2+r}(Y_H - \bar{Y}) & \text{if } Y_2 = Y_H; \\ Y_3 & \text{if } Y_2 = Y_L. \end{cases}$$

[8]Optimal consumption of period 2 and 3 are then

$$C_2^B = \begin{cases} \bar{Y} + \frac{1+r}{2+r}(Y_H - \bar{Y}) & \text{if } Y_2 = Y_H; \\ Y_L & \text{if } Y_2 = Y_L; \end{cases}$$

and

$$C_3^B = \begin{cases} Y_3 + \frac{1}{2+r}(Y_H - \bar{Y}) & \text{if } Y_2 = Y_H; \\ Y_3 & \text{if } Y_2 = Y_L. \end{cases}$$

3.4 Theoretical Implications

Theoretic implications from consumption functions we derived in previous sections are to be provided here. For the algebraic simplicity, we assume that $r = 0$, $\beta = 1$ in the analysis that follows.

3.4.1 Precautionary Saving and Borrowing Constraints

Figure 3.1 depicts the optimal period-1 consumption level C_1 against the initial wealth level W_1 under different environments.[9] First, if neither borrowing nor lending is allowed, consumers consume all their wealth available at this period. The consumption curve is the 45^o line in the figure. On the other hand, if we do not impose any restriction on borrowing and lending, th consumption curve would be the C^N line with slope 1/3 as what the PIH implies. This line intercepts the 45^o line at $W_1 = \bar{Y}$, which means that the optimal consumption can only be achieved through borrowing or lending, with an exception at $W_1 = \bar{Y}$.

C^B is the optimal consumption curve when the borrowing constraints are introduced. This line is depicted from equations (3.10), (3.11), and (3.12) derived in the previous section. As the figure shows, the consumer act as if there were no borrowing constraints when $W_1 > W_H$. Given the plenty wealth this consumer is holding, his optimal consumption plan can be achieved without borrowing from the bond market. That is, the optimal consumption today still leaves 'enough' assets for the next period, which makes the period-2 optimal consumption unconstrained, even if the realized income is Y_L. To the other extreme, for the consumer whose wealth holding is less then W_L, he foresees the higher income in the future and would like to borrow in the bond market to finance his optimal consumption plan. Since borrowing is not possible and the marginal utility is relatively high in this period, he consumes all resources available to him.

With wealth holding between W_L and W_H, the consumer can consume as much as the unconstrained case ($\bar{Y} \leq W_1 \leq W_H$), or as the autarky case ($W_L \leq W_1 < \bar{Y}$). But he chooses to consume less than these two cases. This results from the possibility that he

[9]Similar figure can also be found in Dow and Olsen (1991).

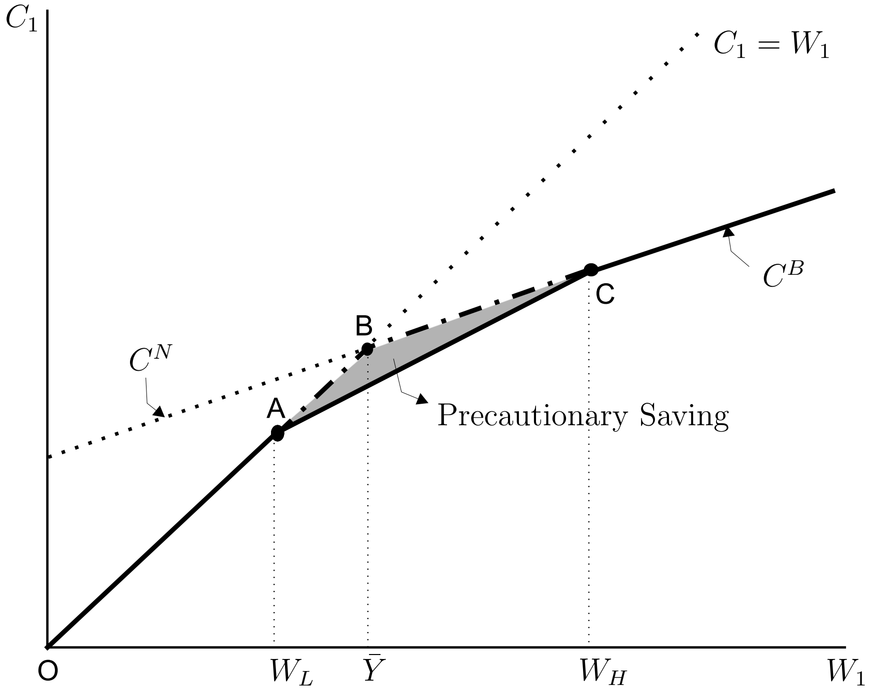

Figure 3.1: Consumption Functions and the Precautionary Saving

will be borrowing constrained in the future. In this case, he will have to sharply reduce his consumption expenditure if he has not accumulated enough assets from the previous periods. His consumption thus fluctuates fiercely, which rise-averse consumers do not wish to experience. Since borrowing is not possible when his optimal consumption demands borrowing, he saves at present to self-finance his needs.

This reduction in consumption stems from the precautionary saving motive. Precautionary saving is defined as the saving that serves to insure against future uncertainties, such as the income shocks. The magnitude of precautionary saving can thus be evaluated by comparing the consumption level between the certain and the uncertain environments. If the future income is known to be $\bar{Y}$ for certain and the borrowing constraints are imposed, those with wealth less than $\bar{Y}$ are contemporaneously constrained. They can consume only out of their wealth holding, as is represented by $\overline{OB}$ in Figure 3.1. For those who are endowed with wealth greater than $\bar{Y}$, they act as what the PIH implies, as $\overrightarrow{BC}$ shown in Figure 3.1.[10] Contrasting between $\overrightarrow{OBC}$ and C^B, we can see that when

[10]Consumption function of this certain environment with borrowing constraint is

$$C_1 = \begin{cases} W_1 & \text{if } W_1 \leq \bar{Y}; \\ \frac{1}{3}(W_1 + 2\bar{Y}) & \text{if } W_1 > \bar{Y}. \end{cases}$$

there are income uncertainties facing consumers, the consumer with medium wealth level chooses to consume less. This reduction in consumption and hence the increase of saving, $\triangle ABC$, is thus the precautionary saving that stems from the presence of the borrowing constraints.

The reason why borrowing constraints can induce the precautionary saving is quite intuitive: to maximize his life-time utility, the consumer substitutes his consumption intertemporally to hedge against future income risk that might result in a very volatile consumption pattern. If there were no borrowing constraints, this saving is not necessary since he can simply borrow in the debt market to smooth out negative income shocks. The question then goes to why this precautionary saving only occurs at the middle wealth region. For the consumer with $W_1 < W_L$, the marginal contribution to the life-time utility of his current consumption is greater than the benefit of saving precautionarily to ensure the smoothness of the consumption. Contrarily, with $W_1 > W_H$, precautionary saving is not necessary because optimal consumption today will leave plenty of assets to the following periods. The effects of the negative income shock can be buffered through this saving, without resorting to the debt market. This finding is consistent with Lusardi (1998), which states that the precautionary saving cannot explain the wealth accumulation behavior of the very rich.

The precautionary saving S^B, is defined as the difference between C^B and $\overrightarrow{OBC}$:

$$S^B = \begin{cases} 0 & \text{if } W_1 \leq W_L \\ \frac{2}{4-p}(W_1 - \bar{Y}) + \left(\frac{1-p}{4-p}\right)(\bar{Y} - Y_L) > 0 & \text{if } W_L < W_1 \leq \bar{Y}; \\ \frac{-2}{3}\left(\frac{1-p}{4-p}\right)(W_1 - \bar{Y}) + \left(\frac{1-p}{4-p}\right)(\bar{Y} - Y_L) > 0 & \text{if } \bar{Y} \leq W_1 < W_H \\ 0 & \text{if } W1 \geq W_H. \end{cases} \tag{3.13}$$

The precautionary saving results from the consumer's concern that he will experience a great reduction in consumption when the realized income is low and no borrowing is allowed. To prevent himself from this volatile consumption pattern, he sacrifices some of his current consumption and thus engages in the precautionary saving behavior. Whether the precautionary saving is conducted is therefore a matter of trading off between the marginal utility loss of reducing current consumption and the marginal benefit from the increased expected period-2 marginal utility from the reduced volatility. When the consumption level is low, such as the case where $W_1 \leq W_L$, the period-1 marginal utility is very high relatively to the possible gain in life-time utility from smoothing consumption. There is no reason to substitute intertemporally if the consumer is to maximize his expected life-time utility. We observe zero precautionary saving as a result.

When the wealth level is above W_L, however, the marginal utility of period-one consumption is now lower than that of the expected period-two consumption. The consumer begins to engage in intertemporal substitution as the marginal rate of intertemporal substitution exceeds one. The precautionary saving begins to rise and reaches its peak at $W_1 = \bar{Y}$. As the expected marginal utility of period-two consumption decreases with the increased period-two consumption, the precautionary saving then decreases as a result.[11] The consumer stops this precautionary saving behavior when his wealth exceeds W_H. This results from the fact that his period-one optimal consumption leaves plenty of assets for the remaining of his life. The presence of the borrowing constraints do not affect his consumption behavior as a result. This finding is also consistent with the empirical findings of Lusardi (1998) that the precautionary saving behavior cannot explain the wealth accumulation behavior of the very rich.

Note that precautionary saving increases with $(\bar{Y} - Y_L)$. The more Y_L deviates from $\bar{Y}$, the the more volatile consumption pattern will be when realized income is Y_L and borrowing opportunity is not available. The precautionary saving motive becomes stronger as a result. Also note that the extent to which Y_H deviates from $\bar{Y}$ does not affect the precautionary saving behavior. This results from the fact that lending is not restricted in our model, as mentioned earlier in the previous section.

[11] The marginal utility function of the quadratic utility is originally a linear one. Its marginal utility thus decreases at a constant rate. When the borrowing constraints are introduced into the model, however, the marginal utility exhibits convexity as what will be proved later in this chapter. The expected utility of period-two consumption therefore no longer decreases at a constant rate.

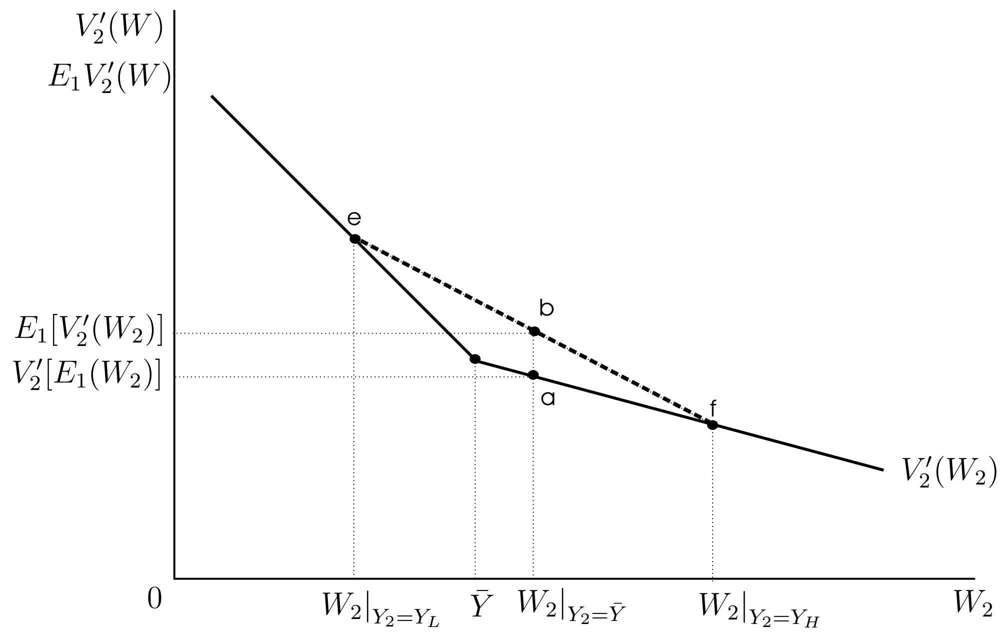

Figure 3.2: $V'(\cdot)$ and Uncertainty

3.4.2 Borrowing Constraints and the Prudence of Value Function

It is well known from Kimball (1990) that, the intensity of the precautionary saving depends on the convexity, or the 'prudence' defined in the paper, of marginal value function. We verify in this subsection that borrowing constraint will boost the convexity of the marginal value function, and the precautionary saving is induced as a result.

When borrowing constraints are absent, marginal value function of the quadratic utility function is a linear one. The magnitude of uncertainty does not affect the expected value of marginal value function. Consumers have no motivation to engage in precautionary saving.[12] We have shown in the previous section, however, that when borrowing constraints are introduced into the model, precautionary motive will be present. We prove here that this simply comes from the prudence of value function induced by borrowing constraints.

The convexity of marginal value function can be explored by investigating its slope.

[12]Optimal consumption requires $U'(C_1) = E_1V_2'(W_2)$. Thus, if $V_2'(\cdot)$ is linear, as quadratic utility function implies, $E_1V_2'(W_2) = V_2'[E_1(W_2)]$, and optimal consumption is not altered by the introduction of uncertainty.

From equation (3.6), the second derivative of the value function is:

$$V_2''(W_2)|_{W_2 \geq \bar{Y}} = -\frac{1}{2} > -1 = V_2''(W_2)|_{W_2 < \bar{Y}} .$$

Marginal value function is now convex with a kink at $W_2 = \bar{Y}$, since its slope is shallower when $W_2 \geq \bar{Y}$, as the $V_2'(W_2)$ in Figure 3.2. This figure also illustrates how this convexity intensify precautionary saving. Suppose that period-2 income is known to be $\bar{Y}$ for certain, and the wealth level corresponding to it is denoted by $W_2|_{Y_2=\bar{Y}}$. $W_2|_{Y_2=\bar{Y}}$ is also equal to the expected period-2 wealth under uncertain environment, denoted $E_1(W_2)$. $V_2'(W_2|_{Y_2=\bar{Y}}) = V_2'[E_1(W_2)]$ is the value of marginal value function associated with this wealth level, as point a in Figure 3.2 implies. When period-2 income is uncertain, with wealth levels $W_2|_{Y_2=Y_L}$ or $W_2|_{Y_2=Y_H}$ corresponding to Y_L and Y_H received by consumers, the expected marginal value will be a convex combination of marginal values associated with each possible wealth level (point e and f, respectively). The expected value of the marginal function depicted in Figure 3.2 is thus point b, a convex combination of e and f, labeled $E_1[V_2'(W_2)]$ in the figure.[13]

Contrasting between point b and point a, it is clear that when income uncertainties are present, the expected marginal value function will be greater than the certain environment counterpart:

$$E_1[V'(W_2)] > V'[E_1(W_2)].$$

Thus we can know from the first-order condition that

$$U'(C_1^B) = E_1[V_2'(W_2)] > V_2'[E_1(W_2)] = U'(C_1^C) \Rightarrow C_1^B < C_1^C,$$

where C_1^C denotes the optimal consumption in the certain environment; C_1^B is the uncertain environment counterpart, as we have defined previously.[14] According to law of diminishing marginal utility, we know that optimal consumption in the model with uncertainties is less than that in the certain environment. This results from the convexity

[13]Point b in Figure 3.2 is plotted under assumptions that $\bar{Y} - Y_L = Y_H - \bar{Y}$, and consumers have equal probabilities to receive Y_H or Y_L in each period. With this setting, the introduction of uncertainty will not affect the expected income, and is thus a mean-zero risk.

[14]C_1^C is the optimal consumption level in the certain environment where borrowing constraints are present. Thus if we connect C_1^C with different wealth levels, we can trace out the consumption function, as $\overrightarrow{OBC}$ depicted in Figure 3.1. Contrarily, if there were no borrowing constraints, optimal consumption function is the C_1^N in Figure 3.1, both in the certain and uncertain environment.

of marginal value function induced by borrowing constraints, and consumers save precautionarily as a result.

Since precautionary saving stems from the consumer's desire to buffer future income uncertainty, we can investigate the impact of this uncertainty on precautionary saving. When income uncertainty is greater, both high and low income are more remote from $\bar{Y}$. Points e and f shift leftwards and rightwards, respectively. Point b shifts upwards, with higher $E_1[V_2'(W_2)]$ corresponding to it. This reveals that when facing greater income uncertainties, consumers consume less at present. The precautionary saving is boosted as a result.

As is defined by Kimball (1990), the prudence of the value function, $-V'''(W)/V''(W)$, measures the convexity of the marginal value function and hence the intensity of precautionary saving. With kinks in marginal value function, this measure is not applicable in our model since the marginal value function is not differentiable at the kink. However, we have proved here that the presence of the borrowing constraints do boost prudence, and hence the precautionary saving.

This result in some way resembles the findings of Carroll (2001a) and Carroll and Kimball (2001).[15] Carroll (2001a), with the numerical analysis technique and the CRRA utility specification, finds that when consumers are impatient and face positive possibility of receiving zero income, their consumption behavior would be very similar to the behavior when they face the possibly binding borrowing constraints. The intuition behind this result is quite straightforward. Foreseeing the possibility of receiving zero income in the future, where the marginal utility of consumption will be vary high, consumers thus leave positive wealth to the next period to prevent themselves from zero consumption. This is equivalent to self-imposed borrowing constraints that restrain themselves form borrowing in each period. On the other hand, Carroll and Kimball (2001) prove that both the presence of the borrowing constraints and the future income uncertainty would lead to concave consumption functions. This would boost the prudence of the value function and

[15]Carroll and Kimball (2001) also prove that consumers will engage in the precautionary saving behavior under the quadratic utility specification. Their model is based on an infinity horizon setup, the analytical form of the precautionary saving thus cannot be provided. Further analysis on the precautionary saving behavior can not be conducted as a result.

thus induces the precautionary saving behavior.

3.4.3 Concavity of Consumption Function

C^B in Figure 3.1 reveals that when borrowing constraints are imposed, the marginal propensity to consume out of current wealth is everywhere greater than or equal to that when borrowing constraints are absent. Consumption thus exhibits excess sensitivity when borrowing constraints are imposed.[16] In addition, among three wealth regions, the slope of consumption function decreases as wealth increases. This means that consumption exhibits concavity, as shown numerically in Zeldes (1989b), Ludvigson (1999), and Carroll (2001a); or empirically in Lusardi (1996), Souleles (1999), and Parker (1999).[17]

Concerning concavity of consumption function, Carroll and Kimball (1996) proves that, *without the presence of liquidity constraints*, the consumption function will be concave in the HARA class of utility functions, with quadratic utility function as its exception. By contrast, we have illustrated here that with the introduction of borrowing constraints, the quadratic utility function will also yield a concave consumption function. This finding is consistent with Carroll and Kimball (2001). They prove first that the concavity of consumption function will boost absolute prudence and hence precautionary saving. Theoretical justifications of the similarity between uncertainty and liquidity constraints are then provided, by showing that they both deliver concave consumption functions. Although some similar conclusions are drawn, with quadratic utility along with 3-period setup, we are able to calculate precautionary saving directly and investigate the concavity of consumption explicitly, which makes our analysis more comprehensive and intuitively understandable.

[16]Here we define excess sensitivity as the difference between the response in consumption and what is implied by PIH, as in Hall and Mishkin (1982).

[17]Lusardi (1998) and Souleles (1999) do not test directly the concavity of consumption. But they do find that the marginal propensity to consume is substantially higher for consumers with low income or wealth. So far as we know, Parker (1999) is the only research that aims to provide empirical evidence on the concavity of consumption function directly. Using non-parametric method in estimating consumption function, he find that consumption function do exhibit concavity when risk is not fully insured.

3.5 Comparative Statics

In previous sections, we have derived analytically the optimal consumption and the induced precautionary when there are borrowing constraints facing consumers. Our study thus has advantages over previous literature in that comparative statics can be undertaken, and factors affecting precautionary saving can be investigated carefully, without resorting to numerical techniques. In an economy where individuals have limited access to borrow against income uncertainties, factors such as future income volatility, expected income, and the probability of being constrained will all affect consumption. How these factors affect consumption via precautionary saving then constitutes main goal of this section.

For algebraic simplicity, we re-parameterize the model as what follows. Let $Y_H - \widetilde{Y} = \widetilde{Y} - Y_L = \delta$, where $\widetilde{Y} = (Y_H + Y_L)/2$. The expected income is then $\bar{Y} = \widetilde{Y} + (2p-1)\delta$; and income volatility is now $Var(Y) = 4p(1-p)\delta^2$. With this setup, changes in δ, $\widetilde{Y}$, and p can be used to discuss future income volatility, expected income, and the probability of being constrained, respectively.

3.5.1 Change in Income Volatility

The effects of the income uncertainty on the consumption behavior have also been investigated in Dow and Olsen (1991). Yet to provide intuitions for the following analysis, these effects are also discussed in detail here. Note that in our re-parameterized model, an increase in δ raises the volatility of income, but $\bar{Y}$ changes simultaneously. To isolate the effect of income volatility, we assume $p = 0.5$. This makes the change in δ a mean-preserving spread, and thus δ governs the volatility of income process, while keeping mean income unaltered.

When borrowing constraints are absent, consumers care only expected income in the quadratic utility setting. Change in δ does not affect C_1^N as a result. When borrowing constraints are imposed, however, this so-called 'certainty equivalence' property no longer remains. With the increase in δ, Y_H increases and Y_L decreases. As mentioned in previous sections, owing to our asymmetric setting that only borrowing is restricted, consumers care only the extent to which Y_L deviates from expected income, because this governs whether consumers consumption level will decline sharply from its 'permanent level'. Thus, change in δ affects consumption and precautionary saving through this deviation of Y_L from $\bar{Y}$.

The effects of income volatility on consumption are twofolds. It affects not only the intensity of precautionary motive, but also who in the economy will engage in the precautionary saving. For the middle-wealth consumers, an increase in the income volatility will translate into a sharp decrease in consumption when low income is realized and borrowing is prohibited. Foreseeing this, consumers reduce current consumption and use the increased saving to insure against this income risk. This can be verified from equation (3.11) that

$$\frac{\partial C_1^B}{\partial \delta} = -\left(\frac{1-p}{4-p}\right) < 0.$$

This reduction in consumption is therefore the precautionary saving, $\frac{\partial S^B}{\partial \delta} = \left(\frac{1-p}{4-p}\right) > 0$.

Besides the increased precautionary saving, the increase in δ also enlarges the wealth region within which individuals engage in precautionary saving. In response to the increase in income volatility that may make consumption fluctuate fiercely, consumers begin to save precautionarily from a lower wealth level. Meanwhile, a higher wealth level is required to make the optimal consumption unconstrained. That is, it demands a higher wealth level to make borrowing constraints and income uncertainties irrelevant. Thus W_L shifts downwards ($\frac{\partial W_L}{\partial \delta} = -\left(\frac{1-p}{2}\right) < 0$), and W_H shifts upwards ($\frac{\partial W_H}{\partial \delta} = \frac{3}{2} > 0$). The middle wealth level becomes larger as a result.

For consumers with extremely high or low initial wealth, their consumption behavior is not altered by changes in income volatility. The very rich can buffer income volatility thorough their own wealth accumulated before. The very poor are constrained by their concurrent borrowing constraints. Their marginal utility to consume today is so high that they do not want to engage in the precautionary saving.

These results are summarized in Figure3.3. Without borrowing constraints, optimal consumption C^N is not affected by income volatility. On the contrary, in an economy with borrowing constraints, the increase in income volatility shifts the consumption curve from

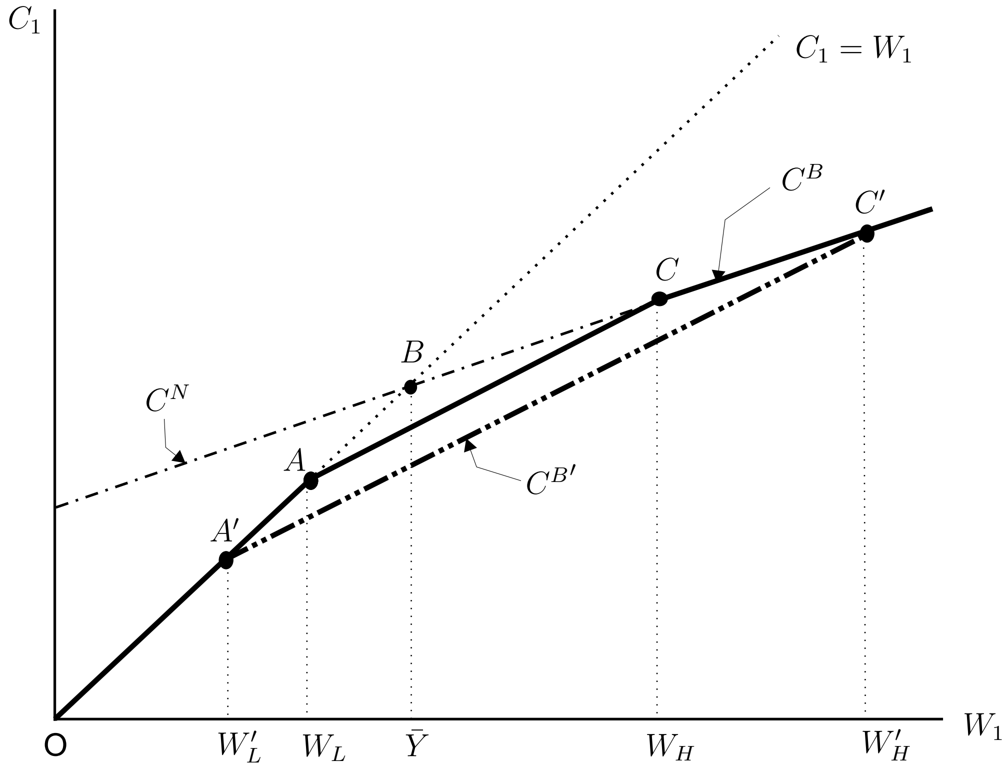

Figure 3.3: Precautionary Saving and Income Volatility

C^B to $C^{B'}$, and the two thresholds shift to W'_L and W'_H, respectively. This then leads the precautionary saving to expand from $\triangle ABC$ to $\triangle A'BC'$ along with the increase in future income uncertainties.

3.5.2 Change in Expected Income

Change in expected future income affects consumer's expected permanent income. The optimal consumption thus responds to this change, whether or not there are borrowing constraints in the economy. What we want to investigate here is that, in responding to the same change in expectation, whether the presence of borrowing constraints alters consumer's reaction to this change. To achieve this, we assume that the change in the expected income comes merely from the change in $\widetilde{Y}$. This leaves the income volatility unaltered when we want to study the effects of the change in the expected income.

We summarize the effects of the change in the expected income in Figure3.4. When there are no borrowing constraints, C^N shifts upwards to $C^{N''}$, with the amount equals to two-thirds of change in $\bar{Y}$. This increment ($\frac{\partial C_1^N}{\partial \widetilde{Y}} = \frac{\partial C_1^N}{\partial \bar{Y}} = \frac{2}{3}$) results form the increase

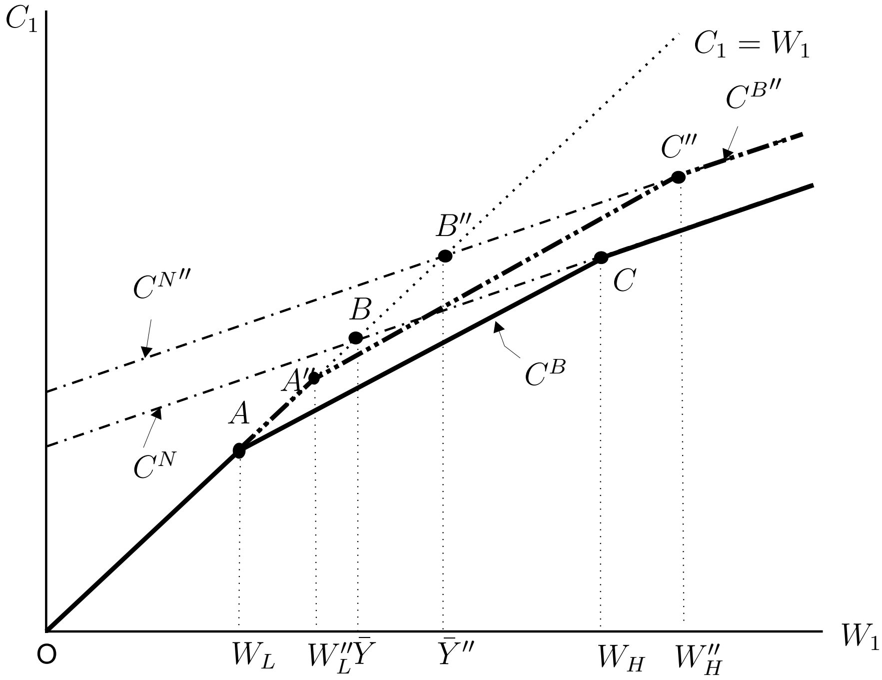

Figure 3.4: Consumption and Change in Expected Income

in the expected period-2 and period-3 income, divided into three periods. In other words, the consumption increases as much as the increase in permanent income, as implied by PIH.

Individuals facing borrowing constraints consume more conservatively. The consumption curve shifts upwards to $C^{B''}$, but the increment of consumption is less than that of the economy where borrowing constraints are absent. This can be verified from the consumption function of the middle-wealth consumers:

$$\frac{\partial C_1^B}{\partial \bar{Y}} = \frac{2}{4-p} < \frac{2}{3} = \frac{\partial C_1^N}{\partial \bar{Y}}.$$

With the same expectation on future income, consumers facing borrowing constraints do not increase as much their consumption. This surely comes from the precautionary motive for buffering income shocks that may make consumption very volatile when consumers are restrained from borrowing opportunities.

From equations (3.8) and (3.9), we know that the two thresholds shift up accordingly:

$$\frac{\partial W_H}{\partial \bar{Y}} = \frac{\partial W_L}{\partial \bar{Y}} = 1.$$

The increase in W_H reveals that a higher wealth level is required to make borrowing constraints irrelevant. That is, consumers with wealth levels between W_H and W''_H (new threshold after the increase of $\bar{Y}$) will switch form being in high wealth level to being the medium wealth consumers who will engage in precautionary saving. The reason is that consumption increases with the increase in expected permanent income. Since current wealth has not changed, this increase in consumption leaves fewer resources available for the following periods. These resources are no longer enough for hedging future income risks. Precautionary saving then becomes a must for rational consumers.

W_L also shifts upwards to W''_L. Consumers with wealth levels between W_L and W''_L now no longer save precautionarily. Because Y_L increase along with the increase in expected income, and thus the 'cost' of being constrained becomes smaller. This reduces the motivation to substitute consumption intertemporally. Consumers then spend all resources available in current consumption that provides relatively high marginal utility.

Corresponding to these two thresholds, those whose wealth level lies in high-wealth region will act as if there were no borrowing constraints. Consumption shifts for 2/3 of the increment in $\widetilde{Y}$. For consumers with low wealth level, their consumption is restricted to wealth available for them. Consumption is not altered by this change in expected income accordingly.

3.5.3 Change in the Probability of Being Constrained

In our model, p denotes the probability of receiving high income in the future. The probability of being borrowing constrained is therefore $1-p$. With the change in p, expected income and the extent to which Y_L deviates form $\bar{Y}$ change as well.[18] This probability then affects consumption through these two channels. More importantly, in addition to influencing consumption through the expected income and income volatility, it also affects consumption via altering higher-order moments of the income distribution, such as the skewness. The reason why skewness may affect consumption is quite intuitive. Consumers facing borrowing constraints dread receiving low income and at the same time being constrained. Other things being equal, consumers thus prefer a income distribution that is

[18] The effects are $\partial\bar{Y}/\partial p = 2\delta$, and $\partial(\bar{Y} - Y_L)/\partial p = 2\delta$, respectively.

more left-skewed. As income now lies more frequently on high level, their consumption is thus constrained less frequently.

Algebraically, when borrowing constraints are present, the effect of change in p on middle-wealth consumers can be divided into three parts:

$$\begin{aligned}\frac{\partial C_1^B}{\partial p} &= \frac{\partial}{\partial p}\left[\bar{Y} + \left(\frac{2-p}{4-p}\right)(W_1 - \bar{Y}) - \left(\frac{1-p}{4-p}\right)(\bar{Y} - Y_L)\right] \\ &= \underbrace{\left(\frac{2}{4-p}\right)2\delta}_{\text{mean}\uparrow} - \underbrace{\left(\frac{1-p}{2-p}\right)2\delta}_{\text{deviation}\uparrow} + \underbrace{\left[\frac{-2}{(4-p)^2}(W_1 - \bar{Y}) + \frac{3p}{(4-p)^2}2\delta\right]}_{\text{left-skewed}} \\ &= \left[\frac{3+8p-p^2}{(4-p)^2}\right]2\delta - \frac{2}{(4-p)^2}(W_1 - \widetilde{Y}). \end{aligned} \tag{3.14}$$

Besides the increased mean and deviation of Y_L from mean, an increase in p also makes the income distribution more left-skewed.[19] This reduces the probability of being borrowing constrained and therefore the intensity of precautionary motive. Consumption increases for $\left[\frac{-2}{(4-p)^2}(W_1 - \bar{Y}) + \frac{3p}{(4-p)^2}2\delta\right]$ (> 0) as a result.[20] This increment in consumption is less than that of the economy without borrowing constraints $\frac{\partial C_1^N}{\partial p} = \frac{4}{3}\delta$. (See appendix for detailed proof.) The reason is that the presence of borrowing constraints makes consumers behave more conservatively.

The wealth region where consumers engage in precautionary saving also changes. Change in W_H stems from the increased mean and deviation:

$$\frac{\partial W_H}{\partial p} = \frac{\partial}{\partial p}\left[\bar{Y} + \frac{3}{2}(\bar{Y} - Y_L)\right] = \underbrace{2\delta}_{\text{mean}\uparrow} + \underbrace{(3/2)2\delta}_{\text{deviation}\uparrow}$$

$$= 5\delta > 0.$$

[19]When p raises, income distribution is more left-skewed:

$$\begin{aligned}\frac{\partial}{\partial p}\left[\frac{E(Y-\bar{Y})^3}{[E(Y-\bar{Y})^2]^{3/2}}\right] &= \frac{\partial}{\partial p}\left[\frac{1-2p}{\sqrt{p(1-p)}}\right] \\ &= -2 - \frac{1}{2}(1-2p)^2 p^{-3/2}(1-p)^{-3/2} < 0,\end{aligned}$$

where $\frac{E(Y-\bar{Y})^3}{[E(Y-\bar{Y})^2]^{3/2}}$ is the coefficient of skewness.

[20]Substituting $W_1 \in [W_L, W_H]$ into it, the effect of the left-skewed income distribution is given by $\left[\frac{-2}{(4-p)^2}(W_1 - \bar{Y}) + \frac{3p}{(4-p)^2}2\delta\right] \in [0, (4-p)2p\delta]$.

W_L also shifts upwards:

$$\frac{\partial W_L}{\partial p} = \frac{\partial}{\partial p}\left[\bar{Y} - \left(\frac{1-p}{2}\right)(\bar{Y} - Y_L)\right] = \underbrace{2\delta}_{\text{mean}\uparrow} - \underbrace{\left(\frac{1-p}{2}\right)2\delta}_{\text{deviation}\uparrow} + \underbrace{\frac{1}{2}(\bar{Y} - Y_L)}_{\text{left-skewed}}$$

$$= (1 + 2p)\delta > 0.$$

Note that W_H is not affected by the change in skewness. When deciding W_H, consumers simply calculate the wealth level that makes their optimal consumption unconstrained even if the realized income is Y_L. The extent of income skewness is therefore beyond consumers' concern.

In Figure3.5.3, W_L''' and W_H''' correspond to two new thresholds after the raise in p. When there are no borrowing constraints, consumption shifts form C^N to $C^{N'''}$, with the increment of $\frac{4}{3}\delta$ from change in expected income. When borrowing constraints are introduced, consumption curves shifts upwards from C^B to $C^{B'''}$. Moreover, with the new expected permanent income $\bar{Y}'''$, precautionary saving shifts from S^B to $S^{B'''}$.

The importance of higher-order moments of income distribution is worth emphasizing here. In addition to discovering the importance of second-order moment of income in a quadratic utility setting, such as Carroll (2001a), we find that as long as there is positive possibility of being constrained in the future, consumers do care about higher-order moments of their income distribution. The importance of higher-order moments has been justified recently by Ludvigson and Paxson (2001), and Carroll (2001b). They argue that in estimating consumption Euler equations, higher-order consumption moments that are regarded as error terms of the linearized Euler equations are all *endogenous*. Ignoring them thus results in many empirical anomalies. This importance of higher-order moments are, however, derived in the CRRA utility setting without borrowing constraints. In this chapter, we have demonstrated that with the imposition of borrowing constraints, higher-order moments may also significantly affect the consumption behavior, even with the quadratic utility setting. This finding, so far as we know, has not been explored in previous literature, and may potentially be able to resolve consumption puzzles that was found empirically.

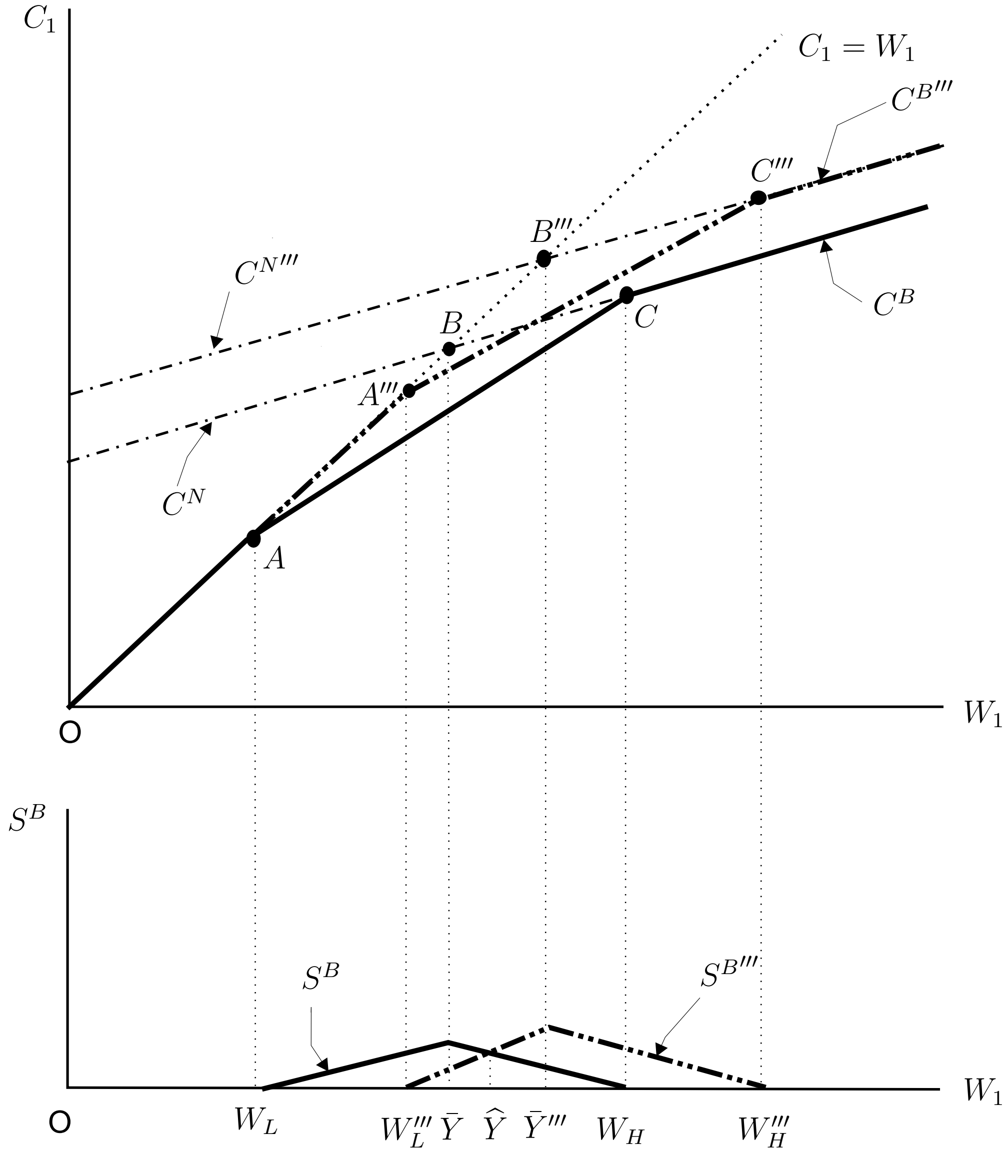

Figure 3.5: Effects of Change in p

3.6 Empirical Implications

Our theoretic work delivers many empirical implications, such as the the causes of excess sensitivity, the consequences of sample splitting between the rich and the poor, as well as the relevance of the higher-order moments to consumption dynamics. We now turn to these issues in this section.

3.6.1 Excess Sensitivity of Consumption

Whether consumption data exhibits excess sensitivity to income has been a mainstream in judging the validity of PIH, since Hall's (1978) seminal work. The presence of borrowing constraints makes the consumption level of the constrained consumer directly link to his current income. This leads consumption sensitive to income, and the presence of borrowing constraints is therefore regarded as an important cause of the empiric failure of the PIH.[21]

In our model, consumption behaviors of the very rich and the very poor have their distinct empirical implications. For consumers whose wealth level lies in the high-wealth region, the presence of borrowing constraints has no essential effect on their consumption behavior. Their consumption dynamics thus follow random walk, as what the standard PIH implies. On the other hand, for consumers who are currently borrowing constrained, their consumption level is directly linked to their current income, and thus exhibit excess sensitivity.[22] This distinct consumption pattern thus serves as the main criterion for judging whether the presence of borrowing constraints is the main cause of the empirical failure of the PIH. Specifically, if the excess sensitivity of consumption only exists in the borrowing constrained subsample, it is often inferred that the presence of borrowing constraints is responsible for the empirical failure of the PIH.

Nevertheless, most consumers do not hold such extreme wealth levels and are therefore the middle-wealth consumers. They perceive that they might be borrowing constrained in

[21] Others include nonseparable utility functions, the presence of nondurable goods, and myopia, see Browning and Lusardi (1996) for detailed discussion.

[22] Similar phenomenon has also been categorized as myopia consumption behavior. (Campbell and Mankiw, 1989, 1990, 1991; Flavin, 1985, 1991; Shea, 1995; Weber, 2000)

the future, and thus engage in the precautionary saving behavior. Variations in consumption of these currently unconstrained consumers can thus be explained by the expected change in income. The consumption dynamics of these consumers thus exhibit excess sensitivity. This means that if we do not properly distinguish between the high- and the middle-wealth consumers in those who are not currently borrowing constrained, the inference that the presence of the borrowing constraints is not the main cause of the empirical failure of the PIH because the excess sensitivity also exists in the unconstrained subsample might be erroneous.

The following example can clearly demonstrate this idea. Suppose that there is a middle-wealth consumer who is not currently constrained. His optimal consumption level is determined by equation (3.11). Suppose also that the realized period-2 income of this consumer is Y_H. His optimal period-2 consumption would neither be constrained.[23] The realized change in consumption of this 'unconstrained' consumer would be:

$$C_2 - C_1 = \underbrace{\frac{(1+r)(1-p)}{(2+r)^2 - p(1+r)} \left(\bar{Y} - W_1\right)}_{[A]} + \underbrace{\frac{(1-p)(r^2+3r+3)}{(2+r)[(2+r)^2 - p(1+r)]} \left(\bar{Y} - Y_L\right)}_{[B]} + \underbrace{\left(\frac{1+r}{2+r}\right)\left(Y_H - \bar{Y}\right)}_{[C]}. \tag{3.15}$$

In the above equation, the change in consumption can be decomposed into three parts. In part $[C]$, $(Y_H) - \bar{Y})$ evaluates the deviation of realized period-2 income from its expected level. It is not contained in the period-one information set, and would not cause excess sensitivity problem consequently. Nevertheless, part $[B]$ and part $[C]$ represent the consumer's expected change in income, and his expected expected variability of income, respectively. They are all known to the consumer when he is deciding how much to consume in period one, and are thus the direct cause of the excess sensitivity. Part

[23]His consumption level would be

$$C_2 = \bar{Y} + \frac{(1+r)^2}{(2+r)^2 - p(1+r)}(W_1 - \bar{Y}) + \frac{(1-p)(1+r)^2}{(2+r)[(2+r)^2 - p(1+r)]}(\bar{Y} - Y_L) + \left(\frac{1+r}{2+r}\right)(Y_H - \bar{Y}).$$

[B] reveals that the realized change in consumption should be positively correlated with expected change in income. Part [C] demonstrates that other things being equal, when a consumer expects a higher uncertainty of his future income, he would be more conservative on his consumption. This leads to a higher increment in his consumption.

The presence of part [B] and part [C] in determining the change in consumption level has important empirical implications. Empirical works that test the validity of the PIH often resort to the following regression:

$$\Delta C_t = \alpha + \beta_1 r_t + \beta_2 \Delta Y_t + \gamma \mathrm{Z_t} + \epsilon_t,$$

where $\mathrm{Z_t}$ is a vector that consists of other variables in the information set. Contrasting with equation (3.15), the influence of part [A] on the change in consumption would be captured by $\beta_2 \Delta Y_t$ in the regression. Part [B] would be explained by variables in $\mathrm{Z_t}$, and part [C] would be included in the error term ϵ_t. This means that if we run the above regression using the consumption data of this 'unconstrained' consumer, we would find that β_2 is significant and that the excess sensitivity of consumption to income is present. Moreover, consumer's expectation of future income volatility would also makes some of γ significant. Their consumption would also be sensitivity to those variables in $\mathrm{Z_t}$ that are correlated with consumer's expectation of future income volatility. This means that if we base on the observation that the excess sensitivity also exists in the unconstrained consumer to infer that the presence of borrowing constraints cannot explain the empirical failure of the PIH, the result might be erroneous.

To be specific, the 'unconstrained consumers' include both the middle-wealth and the high-wealth consumers. Of them only the high-wealth consumers can be free from the possibility of being constrained in the future. For the middle-wealth consumers, however, they foresee that they might be borrowing constrained in the future and thus engage in the precautionary saving behavior. Although they are not currently constrained, the excess sensitivity of their consumption to expected future income can be expected. Consequently, if we do not properly distinguish between these two groups of consumers, we might overestimate the excess sensitivity inherent in the consumption data of the *really unconstrained* consumers.

Based on this finding, we can now retrospect earlier empirical research on the importance of the presence of the borrowing constraints in explaining the empirical failure of the PIH. Taking empirical works on Taiwan's consumption data as examples, Chan and Hu (1996), Chan and Yeh (1998), Huang (1999) and Chen and Hu (2000) all find that using Taiwan's consumption data, the presence of borrowing constraints is not responsible for the rejection of the PIH. Nevertheless, from the above analysis, we suggest that these findings should be interpreted with caution. Chan and Yeh (1998) and Chen and Hu (2000)'s finding that excess sensitivity represents the common phenomenon of the constrained as well as the unconstrianed consumption data might all result from the precautionary saving behavior of the middle-wealth consumers who are not currently constrained. We are therefore rather conservative on the previous conclusions that the presence of the borrowing constraints cannot explain the empirical failure of the PIH. This also supports Huang's (1999) argument that those consumption behavior that seems like myopia one might result from the precautionary saving behavior. We provide here its theoretical underpinning.

3.6.2 Sample Splitting

It is now standard addressing the importance of the borrowing constraints based on sample splitting technique. For research on micro-data, they distinguish between the 'constrained consumers' and the 'unconstrained consumer' (Zeldes, 1989a; Runkle, 1991; Jappelli, et al., 1998; Chan and Yeh, 1998). For researches that base on the macroeconomic time-series data, they tend to separate between time horizons that are 'credit abundant' and those that are 'credit scarce' (Jappelli and Pagano, 1989, 1994; Bachetta and Geralch, 1997; Ludvigson, 1999). If the excess sensitivity only presents in the borrowing-constrained sub-sample, or in periods where the credit condition is more stringent, the presence of the borrowing constraints are then regarded as the direct cause of the empirical failure of the PIH.

The reason for this sample-splitting method is quite clear. From Figure 3.1, it is evident that there are distinct consumption behaviors among different wealth regions.

To justify whether the theory is supported by data, the very first step is therefore to properly separate the sample into sub-samples corresponding to what the theory implies. In separating the sample, most empirical studies use the amount of asset holding as the indicator of whether the consumer is constarined. Besides the data availability and measurement errors problem, the selection of truncating point seems to be quite subjective and hence ad hoc, when the sample splitting method is employed. For example, Zeldes (1989a) chooses households whose wealth holding is less then twice the average income as the constrained sub-sample. This criterion has been criticized by Jappelli, et al. (1998) that a 17% of the households are wrongly classified .

To avoid subjectively selecting the truncating point, Kuo anc Chung (2002) employ the threshold regression technique in this problem. This technique has its advantage in that it lets the truncating point (the 'threshold') to be determined by data, and can thus be free from the critiques of splitting the sample arbitrarily.

Nevertheless, it is clear from Section 5 that the threshold is time-varying and should be determined endogenously by the model. The above-mentioned subjective separation rules separate the sample at a fixed point. The threshold regression technique, while let the data determine the threshold value, is still truncating the sample with that fixed threshold value. These methods are therefore inconsistent with the time-varying threshold value implied by our model.

For the time-varying threshold value, Baccetta and Gerlach (1997) might be helpful in supporting this argument. They use Kalman filter technique in estimating parameter values, and find that the strength of excess sensitivity is time-varying. They attribute this change in excess sensitivity to the liberalization and deregulation of the financial market. Moreover, Chen and Hu (2000) use interest rate spread as the criterion to judge whether the credit market is complete, and the dummy variable is used to characterize periods where the credit market is incomplete. Their method also exhibits the time-varying property.[24] Other than attribute this to the liberalization of the credit market, we

[24]Thy employ HP filter to find the smoothed trend of the interest rate spread. If the period interest spread is greater than this fitted trend value, this period is categorized as the period where the credit market is incomplete. Since the time trend captured by the HP filter is a smoothed time-varying trend, the spread used to separate between the complete and incomplete credit market is not a constant.

offer alternative explanations of this phenomenon. The uncertainties facing consumers, the expected future income, and the possibility of being constrained in the future all endogenously determine the threshold value defined in this paper. Any changes in these factors will then affect the threshold value and hence the amount of people being borrowing constrained and the amount of people who engage in precautionary saving. The extent of excess sensitivity changes as a result.

The regression model that best fits our theoretical framework is the dynamic switching model of Lee and Porter (1984), Hajvanssiliou and Ioannides (1991), Garcia, et al. (1997), and Jappelli, et al. (1998). Realizing the inadequacy of relying on single indicator such as the amount of asset holding in identifying whether a consumer is borrowing constrained, they resort to extrangeous information such as the years of education the household head has received to calculate the 'probability' of this consumer being borrowing constrained. The threshold value thus becomes a stochastic variable and exhibits time-varying property. Our analysis provides ways to endogenize this time-varying threshold value. Based on our model, the threshold value is endogenously determined by the expected income, expected income variability, and even higher-order moments such as the skewness of income distribution. A empirical model that is consistent with our theoretic framework thus needs to accommodate variables that affect these factors. These variables are then used to determine the threshold value and to estimate the probability that a consumer is borrowing constrained.

3.6.3 Higher-Order Moments

Previous comparative static analysis reveals that, though facing linear marginal utility functions, consumers still respond to the expected income volatilities and even higher-order moments of their income distribution. This results from the precautionary saving behavior induced by the presence of borrowing constraints. Consequently, in the economy where consumers cannot rule out the possibility of being borrowing constrained in the future, ignoring these higher-order moments may result in wrong inference or biased

Nevertheless, it is worth mentioning that although this method is using a time-varying sample splitting criterion, the splitting point is still not determined endogenously.

estimates in empirical work.

The importance of higher-order consumption moments has not been investigated until recently. Ludvigson and Paxson (2001) and Carroll (2001b) find that even when the borrowing constraints are absent, log-linearization or second-order approximation to the consumption Euler equation would introduce substantial approximation bias in the CRRA utility specification. The reason is that the omitted higher-order terms are now regarded as error terms in regression, which makes the orthogonal condition between the regressor and the the error term no longer valid. What is even worse, since these higher-order moments are simultaneously and *endogenously* determined with the regressor (the first- or the second-order moment), they are destined to be correlated with each other. This makes the valid instrumental variables unavailable, because any instruments that are capable of explaining the regressor must be correlated with the higher-order terms. Consistent estimation of the structure parameters such as the elasticity of the intertemporal substitution therefore cannot be achieved even if the method of moment estimators are employed. On the other hand, although Attanasio and Low (2004) have demonstrated that, if a long panel of consumption data is used in estimaiton first-order approximation to the consumption Euler can deliver estimates of the structure parameter without too much bias incurred, the required time panel is too long to be affordable in empirical work.

These findings reveal the importance of the higher-order consumption moments. When the income is the only stochastic factor in the model, as what we have set in this chapter, the higher-order consumption moments are directed linked to the higher-order moments of the income distribution. We have illustrated the importance of these higher-moments with the quadratic utility specification when the borrowing constraints are present. With a more general utility specification such as the CRRA utility function accompanied by the presence of the borrowing constraints, the importance of these higher-order terms will be more prominent. This argument is consistent with that of the Courinchas and Parker (2002). They find that the ignorance over the precautionary saving will lead to 'spurious significance' of demographic variables and thus wrong inference.

3.7 Conclusion

With the theoretic framework of Dow and Olsen (1991), we investigate the precautionary saving behavior that is purely induced by the presence of the borrowing constraints. This precautionary saving behavior results from that the presence of the borrowing constraints will boost convexity in the marginal value function that is originally a linear one in the quadratic utility specification. The certainty equivalence property no longer holds and consumers now respond to uncertainties facing them. In addition to these first and second moments, the optimal consumption decision rule now includes higher-order moments such as the skewness of the income distribution. This is in line with the inclusion of higher-order moments of an asset's return in asset pricing such as Harvy and Siddique (2000) and Dittmar (2002).

Empirical implications of the model are also provided. On the excess sensitivity of consumption to income, we find that the currently unconstrained consumers would engage in precautionary saving behavior which cause excess sensitivity if they foresee the possibility of being borrowing constrained in the future. If we base on the finding that the *ex post* unconstrained consumption data also exhibits excess sensitivity to infer that the presence of the borrowing is not the direct cause of the empirical failure of the PIH, the conclusion might be misleading. On the sample splitting between the rich and the poor, we argue that the thresholds in separating the sample are time-varying and should be determined endogenouly by the model. According to the model, these threshold values may be affected by changes in consumers' expectation on their future income, income volatility, as well aste higher-order moments of their income distribution. How to parameterize these characteristics is thus important if we want to make correct inference on the consumption behavior. Finally, we demonstrate the importance of higher-order moments in consumption decision. This can then be served as the theoretical underpinning of our previous two chapters.

Our model also sheds light on the asset pricing. The consumption-based capital asset pricing model (CCAPM) has been the mainstream in pricing assets since Lucas (1978). In the CCAPM, the marginal rate of intertemporal substitution is used to price all assets.

We have demonstrated in this chapter that with the presence of the borrowing constraints, the consumption dynamics is entirely different from the original PIH one. This leads to a brand new intertemporal substitution behavior and thus a different pricing kernel. In economies where most consumers are not free from the possibility of being borrowing constrained in the future, whether this new consumption dynamics implied by our model can explain the many puzzles that have been found in the asset pricing thus worth further investigation.