8 APPENDIX

The investor's prior distribution over the initial value of β is assumed to be Gaussian, and β is assumed to follow equation (8). Because $de(t)/e(t)$ and $dp(t)$ are following a joint Brownian motion; and all the parameters in equation (4), (8), and (9) are linear functions of the unobservable state variable β, the distribution function of β at time t, $F_t(x) = P(\beta \leq x|\mathcal{F}_t^I)$, is Gaussian given the investor's information structure at time t, $\mathcal{F}_t^I$, which is generated by the joint processes $I(t) = (e(t), P(t))$. Let $b_t \equiv E(\beta|\mathcal{F}_t^I)$; then b will be the optimal estimate of β from $I(t)$. The knowledge of the variance $v_t \equiv E((\beta - b_t)(\beta - b_t)^\top|\mathcal{F}_t^I)$, which is an $n \times n$ matrix, gives us a measure of filtering error and the investor's level of uncertainty. We call the process for $I(t) = (e(t), P(t))$ the investor's signal used to form expectations of β. Let $Z_1 = (Z_1(t), \mathcal{F}_t)$ and $Z_2 = (Z_2(t), \mathcal{F}_t)$ be two mutually independent Wiener processes, where $Z_1(t) = \left[Z_{\beta_1}(t), Z_{\beta_2}(t)\right]$ and $Z_2(t) = \left[Z_{r_d}(t), Z_{r_f}(t), Z_e(t)\right]$. We write the processes for signals as

$$dI = (I_0(I.t) + I_1(I,t)\beta)dt + \sum_{i=1}^{2} C_i(I,t)dZ_i,$$

where $I_0(I,t)$ is an $(2+1) \times 1$ vector with $\overline{\mu}$ in the first row and A_0 in the remaining rows, and $I_1(I,t)$ and $C_1(I,t)$ and $C_2(I,t)$ and matrices of dimension $((2+1) \times 2)$ and $((2+1) \times 2)$ and (3×3), with $L - \overline{L}$ and A_1 forming the rows of the first and zero matrix of the second and.σ_e and σ_p forming the rows of the third. Rewrite the process for β as

$$d\beta = (B_0(I,t) + B_1(I,t)\beta)dt + \sum_{i=1}^{2} c_i(I,t)dZ_i.$$

Define $\Gamma(I,t) = (c \circ C)(I,t)$ as the covariance between the signals and the state variables, $\Sigma(I,t) = (c \circ c)(I,t)$ as the variance–covariance matrix of the state variables, and $\Delta(I,t) = (C \circ C)(I,t)$ as the variance–covariance matrix of the signal processes where $c \circ c = c_1 c_1^\intercal + c_2 c_2^\intercal$, $c \circ C = c_1 C_1^\intercal + c_2 C_2^\intercal$, $C \circ C = C_1 C_1^\intercal + C_2 C_2^\intercal$. A direct extension of Theorem 12.3 in Liptser and Shiryayev (2001) to vectors of measurement and transition equations yields

$$\begin{aligned} db(t) &= [B_0(I,t) + B_1(I,t)b]\,dt + [v(t)I_1(I,t)^\intercal + \Gamma(I,t)]\,\Delta(I,t)^{-1} \\ &\quad \times [dI - (I_0(I.t) + I_1(I,t)b)dt]\,, \end{aligned}$$

$$\begin{aligned} dv(t) &= B_1(I,t)v(t) + v(t)B_1(I,t)^\intercal + \Sigma(I,t) - \\ &\quad [v(t)I_1(I,t)^\intercal + \Gamma(I,t)]\,\Delta(I,t)^{-1}\,[v(t)I_1(I,t)^\intercal + \Gamma(I,t)]^\intercal\,, \end{aligned}$$

where

$$\begin{aligned} d\widehat{W}(t) &= C_2(I,t)^{-1}\left\{dI - E(dI|F_t^I)\right\} \\ &= C_2(I,t)^{-1}\left\{dI - [(I_0(I.t) + I_1(I,t)b)dt]\right\}. \end{aligned}$$

Then we can write the process $I(t)$ as

$$\begin{pmatrix} dr_d(t) \\ dr_f(t) \\ \frac{de(t)}{e(t)} \end{pmatrix} = \left(\begin{pmatrix} a_d(b_d - r_d(t)) \\ a_f(b_f - r_f(t)) \\ \overline{\mu}_e \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ r_d(t) - b_d & r_f(t) - b_f \end{pmatrix} \begin{pmatrix} \beta_1(t) \\ \beta_2(t) \end{pmatrix} \right) dt$$

$$+ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} dZ_{\beta_1}(t) \\ dZ_{\beta_2}(t) \end{pmatrix} + \begin{pmatrix} \sigma_{r_d} & 0 & 0 \\ 0 & \sigma_{r_f} & 0 \\ 0 & 0 & \sigma_e(t) \end{pmatrix} \begin{pmatrix} dZ_d(t) \\ dZ_f(t) \\ dZ_e(t) \end{pmatrix},$$

and β as

$$\begin{pmatrix} d\beta_1(t) \\ d\beta_2(t) \end{pmatrix} = \begin{pmatrix} \vartheta_1(\overline{\beta}_1 - \beta_1(t)) \\ \vartheta_2(\overline{\beta}_2 - \beta_2(t)) \end{pmatrix} dt + \begin{pmatrix} \sigma_{\beta_1} & 0 \\ 0 & \sigma_{\beta_2} \end{pmatrix} \begin{pmatrix} dZ_{\beta_1}(t) \\ dZ_{\beta_2}(t) \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} dZ_d(t) \\ dZ_f(t) \\ dZ_e(t) \end{pmatrix}.$$

The process b and v will be the following equation

$$\begin{pmatrix} db_1(t) \\ db_2(t) \end{pmatrix} = \begin{pmatrix} \vartheta_1(\overline{b_1} - b_1(t)) \\ \vartheta_2(\overline{b_2} - b_2(t)) \end{pmatrix} dt + \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} v_{11}(t) & v_{21}(t) \\ v_{12}(t) & v_{22}(t) \end{pmatrix} \begin{pmatrix} 0 & 0 & r_d(t) - b_d \\ 0 & 0 & r_f(t) - b_f \end{pmatrix} \right]$$

$$\begin{bmatrix} \sigma_{r_d}^2 & 0 & 0 \\ 0 & \sigma_{r_f}^2 & 0 \\ 0 & 0 & \sigma_e^2(t) \end{bmatrix}^{-1}$$

$$\left[\begin{pmatrix} dr_d(t) \\ dr_f(t) \\ \frac{de(t)}{e(t)} \end{pmatrix} - \left(\begin{pmatrix} a_d(b_d - r_d(t)) \\ a_f(b_f - r_f(t)) \\ \overline{\mu}_e \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ r_d(t) - b_d & r_f(t) - b_f \end{pmatrix} \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix} \right) dt \right]$$

$$\begin{pmatrix} dv_{11}(t) & dv_{21}(t) \\ dv_{12}(t) & dv_{22}(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \sigma_{\beta_1}^2 & 0 \\ 0 & \sigma_{\beta_2}^2 \end{pmatrix}$$

$$- \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} v_{11}(t) & v_{21}(t) \\ v_{12}(t) & v_{22}(t) \end{pmatrix} \begin{pmatrix} 0 & 0 & r_d(t) - b_d \\ 0 & 0 & r_f(t) - b_f \end{pmatrix} \right]$$

$$\times \begin{bmatrix} \sigma_{r_d}^2 & 0 & 0 \\ 0 & \sigma_{r_f}^2 & 0 \\ 0 & 0 & \sigma_e^2(t) \end{bmatrix}^{-1}$$

$$\times \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} v_{11}(t) & v_{21}(t) \\ v_{12}(t) & v_{22}(t) \end{pmatrix} \begin{pmatrix} 0 & 0 & r_d(t) - b_d \\ 0 & 0 & r_f(t) - b_f \end{pmatrix} \right]^\intercal$$

where

$$
\begin{pmatrix} d\widehat{Z_d}(t) \\ d\widehat{Z_f}(t) \\ d\widehat{Z_e}(t) \end{pmatrix} = \begin{pmatrix} \sigma_{r_d} & 0 & 0 \\ 0 & \sigma_{r_f} & 0 \\ 0 & 0 & \sigma_e(t) \end{pmatrix}^{-1}
\left[\begin{pmatrix} dr_d(t) \\ dr_f(t) \\ \frac{de(t)}{e(t)} \end{pmatrix} - \left(\begin{pmatrix} a_d(b_d - r_d(t)) \\ a_f(b_f - r_f(t)) \\ \overline{\mu}_e \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ r_d(t) - b_d & r_f(t) - b_f \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right) dt \right].
$$