5 The Optimization Program

Our problem is the selection problem for an optimal self-financing portfolio strategy which maximizes the expected terminal utility. We assume further that the insurer's horizon T is shorter than the maturing dates of the domestic and foreign bonds, which ensures that all bonds are long-lived assets from the insurer's viewpoint. Here we choose the *CRRA* utility function $U(W)$ such as

$$
\begin{aligned}
U(W) &= \frac{1}{\gamma}W^{\gamma},\ 0<\gamma<1 \\
&= \ln W,\ \gamma = 0.
\end{aligned}
$$

The power utility is chosen for two reasons. First, pension funds are in general large companies which define their strategies with respect to the amount of money they are managing, more or less in a scaling way. This feature is well captured by the use of the power utility function. Second, pension funds are regulated in such a way that they can not reach negative values. This is true also in the power utility case, thanks to the infinite marginal utility at zero.

The wealth $W(t)$ of the investor at each time t is

$$
\begin{aligned}
W(t) &= \Gamma_{M_d}(t)M_d(t) + \Gamma_{\widehat{M_f}}(t))\widehat{M}_f(t) + \Gamma_{B_d}(t)B_d(t) \\
&\quad +\Gamma_{\widehat{B_f}}(t))\widehat{B_f}(t) + \Gamma_{S_d}(t)S_d(t) + \Gamma_{\widehat{S_f}}(t))\widehat{S_f}(t),
\end{aligned}
$$

where $(\Gamma_i(t) : i \in \{M_d, \widehat{M_f}, B_d, \widehat{B_f}, S_d, \widehat{S_f}\})$ stand for the numbers of units of each asset. Applying Ito's lemma under the consideration of self-financing strategy and noting that the domestic money market account is a riskless asset from the insurer's viewpoint, we have (c.f. Merton (1971))

$$\frac{dW(t)}{W(t)} = (\cdot)dt + \pi(t)^{\intercal}\Theta(t)dZ(t), \tag{10}$$

where

$$\pi(t)^{\intercal} = \left[\pi^{\widehat{M_f}}(t)\ \pi^{B_d}(t)\ \pi^{\widehat{B_f}}(t)\ \pi^{S_d}(t)\ \pi^{\widehat{S_f}}(t)\right],$$

is the portfolio weight vector of the risky assets and $(\cdot)$ denotes an irrelevant function, a notation which will be frequently used in the sequel.

Define the optimal growth portfolio $\rho(t)$ as (also see Merton (1992) and Long (1990))

$$\rho(t) = M_d(t)\delta(t)^{-1},$$

then

$$\rho(t) = \exp\left\{\int_0^t \Phi(\tau)^{\intercal} dW(\tau) + \int_0^t \left(r_d(\tau) + \frac{1}{2}\Phi(\tau)^{\intercal}\Phi(\tau)\right) d\tau\right\}.$$

The investor's international portfolio selection problem is written as

$$\max E\left[U(W(T))\right], \quad 0 < \gamma < 1$$

with the martingale constraint

$$E\left[\frac{W(T)}{\rho(T)}\right] = W(0).$$

Here $E[\cdot]$ is the expectation operator under the historical probability measure P. Following Lioui and Poncet (2003) and according to Cox and Huang (1989, 1991), the first order condition of the optimization problem is

$$W(T) = \lambda^{\frac{1}{\gamma-1}} \rho(T)^{\frac{1}{1-\gamma}},$$

where the Lagrange multiplier λ is characterized by

$$W(0) = \lambda^{\frac{1}{\gamma-1}} E\left[\rho(T)^{\frac{\gamma}{1-\gamma}}\right].$$

The optimal wealth $V(t)$ at time t is equal to

$$\begin{aligned} V(t) &= \lambda^{\frac{1}{\gamma-1}} \rho(t) E_t\left[\rho(T)^{\frac{\gamma}{1-\gamma}}\right] \\ &= \lambda^{\frac{1}{\gamma-1}} \rho(t)^{\frac{\gamma}{1-\gamma}} B_d(t,T)^{\frac{\gamma}{\gamma-1}} E_t\left[\theta(t,T)^{\frac{\gamma}{\gamma-1}}\right], \end{aligned} \tag{11}$$

where

$$\theta(t,T) = \frac{B_d(T,T)\rho(t)}{B_d(t,T)\rho(T)} = \frac{\rho(t)}{B_d(t,T)\rho(T)}, \tag{12}$$

and $E_t[\cdot]$ is the expectation operator under the probability measure P and conditional with respect to $\mathcal{F}_t$, the filtration at time t. Defining $E_t\left[\theta(t,T)^{\frac{\gamma}{\gamma-1}}\right]$ as $J(\gamma; t, T)$ and invoking

Itô's lemma, we have formally

$$\frac{dJ(\gamma;t,T)}{J(\gamma;t,T)} = (\cdot)dt + \sigma_J(\gamma;t,T)^\intercal dZ(t),$$

where $\sigma_J(\gamma;t,T)$ is the 5×1 diffusion vector of the process $dJ(\gamma;t,T)/J(\gamma;t,T)$. Applying Itô's lemma to Eq.(11), we have

$$\frac{dV(t)}{V(t)} = (\cdot)dt + \left[\frac{1}{1-\gamma}\Phi(t)^\intercal - \frac{\gamma}{1-\gamma}\sigma_{B_d}(t,T)^\intercal + \sigma_J(\gamma;t,T)^\intercal\right] dZ(t), \tag{13}$$

where

$$\sigma_{B_d}(t,T)^\intercal = \left[\begin{array}{ccccc} 0 & \sigma_{B_d}(t,T_d) & 0 & 0 & 0 \end{array}\right]. \tag{14}$$

Identifying the diffusion terms of (10) and (13) , we obtain the expression of optimal allocation strategy (through the dynamic fund separation theorems) $\pi(t)$ of risky assets as

$$\pi(t) = \Theta(t)^{-1}\left\{\frac{1}{1-\gamma}\Phi(t) - \frac{\gamma}{1-\gamma}\sigma_{B_d}(t,T) + \sigma_J(\gamma;t,T)\right\}. \tag{15}$$

Lastly, turning to the benchmark case of the logarithmic utility, $\Theta(t)^{-1}\left(\frac{1}{1-\gamma}\Phi(t)\right)$ in equation (15) readily reveals the Bernoulli investor's myopic behavior, i.e., the speculative component or optimal growth portfolio. Since prices in the financial market change continuously, the optimal growth portfolio must be rebalanced continuously in order to maintain the proposed weights.

Comparing with the previous research in Lioui and Poncet (2003), we consider the learning effect on the exchange rate movements based on the information collected from the interest rates. The significant difference between our results and their works are the first entry in Eq (15). Since the learning processes influence the premium of exchange rate movements, the crucial changes lie in the difference of market price of risk of the interest rate movements to the updated exchange rates. The constructed optimal investment strategy is influenced by the adjusted factor. Hence the investors should dynamically rebalance their holding portfolio according to the filtering mechanism. The second and third entry in Eq (15) are the same compared to the previous results in Lioui and Poncet (2003), two hedging terms should be considered against two specific sources of risk: The interest rate risk related to the random evolution of the money market account value $M(t)$ accruing at the stochastic rate $r(t)$ and the risk associated with the random fluctuations of the market price of risk $\Phi(t)$ that are embedded in the Radon–Nikodym derivative $\delta(t)$.