Chapter 2

Main Result

It is well known that caterpillars have graceful labellings. In particular, caterpillars have *up/down* labellings [21]. To generalize graceful labellings of caterpillars, Bermond [3] conjectured that lobsters are graceful which is not yet known.
Recently, Chen and Shih [12] proved that 2-caterpillars below, a class of lobsters, are graceful. To generalize Chen-Shih's result, in this Section we try to look for graceful labellings of 4-caterpillars which are defined as follows:

Definition 2.0.1 *An $n-$caterpillar is a tree having a single path only incident to the end-vertices of paths of length $n-1$. The single path is called the $n-$caterpillar's body and those paths attached to its body are called the $n-$caterpillar's legs.*

We herein wish to complete the following result.

Theorem 2.0.2 *4-Caterpillars have graceful labellings.*

For this object, we first provide an algorithm to yield graceful labellings of $4-$caterpillars. The main technique is to deal with 4-caterpillars which have body of length divisible by 4.

Algorithm A: a labelling of any $4-$caterpillar.

Set $\mathbb{T}_4$ to be the class of 4-caterpillars with a body of length divisible by 4. Let T be a 4-caterpillar.

1. Assume that $T \in \mathbb{T}_4$ has $4n+1$ vertices, i.e. the length of body is a multiple of 4. Partition $[4n]$ into n 4-sets $X_i = \{4n-4(i-1), 1+4(i-1), 4n-4(i-1)-1, 2+4(i-1)\}$ for $i = 1, 2, ..., n$.

$$\begin{array}{ccccccc} 4n & 4n-4 & 4n-8 & 4n-12 & \cdots & 8 & 4 \\ 1 & 5 & 9 & 13 & \cdots & 4n-7 & 4n-3 \\ 4n-1 & 4n-5 & 4n-9 & 4n-13 & \cdots & 7 & 3 \\ 2 & 6 & 10 & 14 & \cdots & 4n-6 & 4n-2 \\ X_1 & X_2 & X_3 & X_4 & \cdots & X_{n-1} & X_n \end{array}$$

   (a) Remove all legs of the $(4i-2)^{nd}, (4i-1)^{st}$, and $(4i)^{th}$, vertices of the body to be incident to the $(4i+1)^{st}$ vertex of the body for $i \geq 1$.

   (b) Partition T into a union of $4-$stars ($4-$caterpillars whose body is a single vertex) such that each $(4i+1)^{st}$ vertex (except the first one) of the body is the last leaf of a $4-$star and the root of next $4-$star.

   (c) Assign 0 to the first vertex of the body.

   (d) In $(2k+1)^{st}$ 4-star for $k \geq 0$, choose X_i where i is the unused minima in $[n]$ and orderly label the vertices of legs with the numbers in X_i..

   (e) In $(2k)^{th}$ 4-star for $k \geq 1$, choose X_i where i is the unused maxima in $[n]$ and orderly label the vertices of legs with the numbers reverse in X_i..

2. Assume that $T \notin \mathbb{T}_4$ has $4n + j$ vertices for $j \in \{2, 3, 4\}$.

   (a) Remove all legs incident to the first $j - 1$ vertices of the body to the j^{th} vertex.

   (b) Let the first $j - 1$ vertices be $u_1, u_2, ..., u_{j-1}$. Label $u_{j-1}, u_{j-2}, ..., u_1$ according to the sequence $4n + 1, -1, 4n + 2$.

   (c) Apply step 1 to the remaining 4−caterpillar.

   (d) Replace each label s with $s + 1$ if $j \geq 3$.

3. Restore each removed leg and relabel it according to the following rules: Let $y_i, 4n + 1 - y_i, y_i - 1$, and $4n + 2 - y_i$ be the vertex labels of the leg before restoring.

   (a) If the leg is removed one place, then we relabel these four vertices with $4n + 1 - y_i, y_i, 4n + 2 - y_i, y_i - 1$.

   (b) If the leg is removed two places, then we relabel these four vertices with $y_i - 1, 4n + 2 - y_i, y_i, 4n + 1 - y_i,$.

   (c) If the leg is removed three places, then we relabel these four vertices with $4n + 2 - y_i, y_i - 1, 4n + 1 - y_i, y_i,$.

Let $T \in \mathbb{T}_4$ where its first vertex of the body has at least one leg and it is labelled by algorithm A. We increase the first leg of the first vertex of the body, and replace vertex label s with $s + 4$ in each even vertex of the leg in $(2i + 1)^{th}$−star, and in each odd vertex of the leg in $(2i)^{th}$-star for $i \geq 1$. Let the new labelled 4−caterpillar be T''. Then we have the following result.

Lemma 2.0.3 *Let T'' be constructed as above. Then T'' is labelled by algorithm A.*

Proof. Assume that T has $4n+1$ vertices and is labelled by algorithm A. Following the step 1 of algorithm A, if we increase the first leg of the first vertex of the body, then in the remaining labelled 4-caterpillar, the vertex labels of legs can be

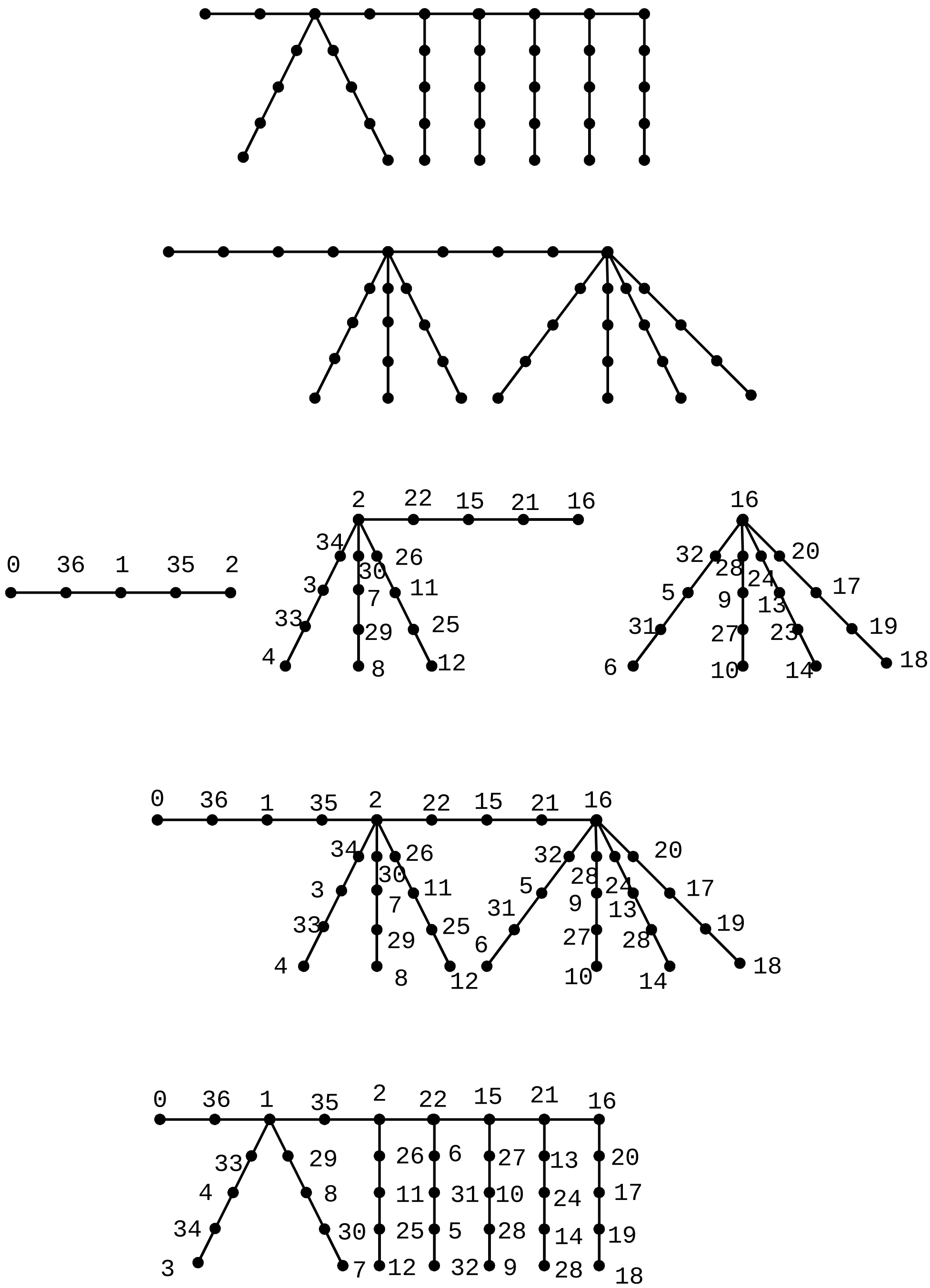

Figure 2.1: A graceful labelling of a 4-caterpillar with a body of length divisible by 4.

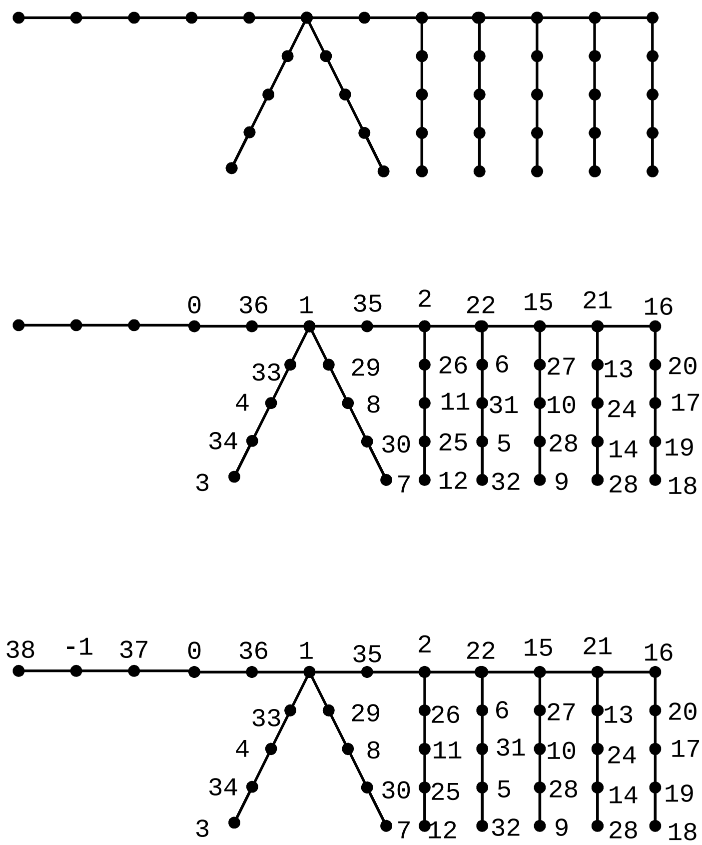

Figure 2.2: A graceful labelling of a 4-caterpillar with a body of length not divisible by 4.

viewed as the following patterns:

$4n$	$4n-4$	$\cdots$	8	4
1	5	$\cdots$	$4n-7$	$4n-3$
$4n-1$	$4n-5$	$\cdots$	7	3
2	6	$\cdots$	$4n-6$	$4n-2$

It corresponds to replace each number s with $s+4$ in the 2^{nd} row and 4^{th} row of the above tabler that we replace vertex label s with $s+4$ in each even vertex of the leg in $(2i+1)^{th}$ 4-star, and in each odd vertex of the leg in $(2i)^{th}$ 4-star for $i \geq 1$. Then the above tabler will be changed into the following patterns:

$4n$	$4n-4$	$\cdots$	8	4
5	9	$\cdots$	$4n-3$	$4n+1$
$4n-1$	$4n-5$	$\cdots$	7	3
6	10	$\cdots$	$4n-2$	$4n+2$

This tabler may be viewed as the vertex labels of legs in $T''$and it is just a labelling of T'' by algorithm A. Hence we complete the proof. ❑

Lemma 2.0.4 *Let*

$$F_1 = \{|x-y| : x \geq 2 \textit{ is even, } y \geq 1 \textit{ is odd } \textit{ and } x+y = 4n+1\},$$

$$F_2 = \{|x-y| : x, y \geq 1 \textit{ are both odd and } x+y = 4n\},$$

$$F_1' = \{|x'-y'| : x' \geq 4 \textit{ is even, } y' \geq 3 \textit{ is odd and } x'+y' = 4n+5\},$$

$$F_2' = \{|x'-y'| : x', y' \geq 3 \textit{ are both odd and } x'+y' = 4n+4\}.$$

Then $F_1 = F_1'$ *and* $F_2 = F_2'$

Proof. On the one hand, $\forall x, y$ satisfying the conditions in the set F_1, if we set $x' = x+2 \geq 4$ and $y' = y+2 \geq 3$,then x' and y' satisfies the conditions in the set F_1' and $|x'-y'|$=$|x+2-(y+2)| = |x-y|$. This implies that $F_1 \subseteq F_1'$.

On the other hand, $\forall x', y'$ satisfying the conditions in the set F_1', if we set $x = x'-2$ and $y = y'-2$, then x and y satisfies the conditions in the set F_1 and $|x-y| = |x'-2-(y'-2)| = |x'-y'|$. This implies that $F_1' \subseteq F_1$. Hence we obtain $F_1 = F_1'$.

Similarly, $F_2 = F_2'$. Then we complete this proof. ❑

Lemma 2.0.5 *If the body of $T \in \mathbb{T}_4$ has no legs except in the $4i+1^{st}$ vertex , then algorithm A yields a graceful labelling of T and the first vertex label of the body is 0.*

Proof. Assume that each vertex, except the $4i+1^{st}$ vertex, of the body of T, with $4n+1$ vertices, has no legs. At the step 1 of algorithm A we know that all vertex labels of T are different. It suffices to prove that the set of edge labels on T by algorithm A is $[4n]$.

We use induction on n. For $n=1$, no mater T is a 4-caterpillar with the body of length 4 or a 4-caterpillar with the body of length 0 and one leg, it is a P_5 which has a graceful labelling 0,4,1,3 and 2 on its vertices in order. Suppose that for any $T \in \mathbb{T}_4$ with $4k+1$ vertices, where each vertex except the $4i+1^{st}$ vertex of the body has no legs, always has a graceful labelling by algorithm A.

Consider a T in $\mathbb{T}_4$ with $4k+5$ vertices, where each vertex except the $4i+1^{st}$ vertex of the body has no legs.

Case 1: The first vertex of the body in T has no legs (see Figure 2.3). Partition T into two parts P_5 and T', where P_5 is a labelled path with 5 vertices and T' is a labelled $4-$caterpillar with $4k+1$ vertices. If we let each vertex label add 2 in T', then there yields a new labelled $4-$caterpillar T". Combining P_5 and T", where the first vertex label of the body is 0 and the other vertices are labelled by algorithm A. By induction hypothesis, T has a graceful labelling whose first vertex label of the body is 0.

Case 2: The first vertex of the body in T has at least one leg (see Figure 2.4). Partition T into two parts P_5 and T', where P_5 is a labelled path with 5 vertices reversely and T' is a graceful labelled by algorithm A with $4k+1$ vertices.

Since the set of vertex labels in P_5 is $\{0, 4n+4, 1, 4n+3, 2\}$, then the set of edge labels is $\{4n+4, 4n+3, 4n+2, 4n+1\}$. The remaining is to show that the set of edge labels in T' is $\{1, 2, ..., 4n\}$. We shall use Lemma 2.0.3 to finish this work. In T', we replace vertex label s of u with $s+4$, if u is an even vertex of legs in $2i+1^{th}$ 4-star or an odd vertex of legs in $2i+2^{th}$ 4-star for $i \geq 0$. By Lemma 2.0.3, there yields a new labelled 4−caterpillar T". Combining P_5 and T",where the first vertex of the body is labelled 0 and the other vertices are labelled by algorithm A. By induction hypothesis, T has a graceful labelling and the set of edge labels in T is $[4k+4]$ which equals to $\bigcup_{i=1}^{4} S_i''$, where

$$S_i'' = \{s : s \text{ is the} i^{th} \text{ edge label in each 4-star}\}$$

for $i = 1, 2, 3, 4$. In fact,

$$S_2'' \cup S_4'' = \{|x-y| : x \text{ is even, } y \text{ is odd, and } x+y = 4n+1\}, \text{and}$$

$$S_3'' = \{|x-y| : x, y \text{ are both odd and } x+y = 4n\}.$$

Let the set of edge labels of T' equal to $\bigcup_{i=1}^{4} S_i'$, where

$$S_i' = \{s : s \text{ is the} i^{th} \text{ edge label in each 4-star}\}$$

for $i = 1, 2, 3, 4$. By Lemma 2.0.4, $S_2'' \cup S_4'' = S_2' \cup S_4'$ and $S_3'' = S_3'$. Since $S_1'' = S_1'$, we obtain $\bigcup_{i=1}^{4} S_i' = [4k]$. Combining this with the edge labels $4k+1, 4k+2, 4k+3, 4k+4$ in P_5, we obtain T has a graceful labelling and the first vertex label of the body is 0. ❑

Now we are in position to prove our main result.

Proof of Theorem 2.0.2. Two cases are discussed.

Case 1:. Assume that T is in $\mathbb{T}_4$. Step 1a of algorithm A makes T be a new T' such that each vertex of the body has no legs except in the $4i+1$ vertex. By Lemma 2.0.5, T' has a graceful labelling. The remaining is to prove after restoring each removed leg, T still has a graceful labelling. Let the vertex labels of the leg before removed are $y, 4n+1-y, y-1$, and $4n+2-y$. There are three subcases as follows:

Subcase 1: The leg is removed one place forward. We orderly relabel four vertices with $4n+1-y, y, 4n+2-y$, and $y-1$. In Figure 2.5(a), before restoring the removed leg, four edge labels in the removed leg are $|x-y|$, $|4n+1-2y|$, $|4n+2-2y|$, and $|4n+3-2y|$. After restoring the removed leg, the new four edge labels in the removed leg are $|x-y|, |4n+1-2y|, |4n+2-2y|$, and $|4n+3-2y|$, i.e. T still has a graceful labelling, where the first vertex label of the body is 0.

Subcase 2: The leg is removed two places forward. We orderly relabel four vertices with $y-1, 4n+2-y, y$, and $4n+1-y$. In Figure 2.5(b), before restoring the removed leg, four edge labels in the removed leg are $|x-y|$, $|4n+1-2y|$, $|4n+2-2y|$, and $|4n+3-2y|$. After restoring the removed leg, the new four edge labels in the removed leg are $|x-y|, |4n+3-2y|, |4n+2-2y|$, and $|4n+1-2y|$, i.e. T still has a graceful labelling, where the first vertex label of the body is 0.

Subcase 3: The leg is removed three places forward. We orderly relabel four vertices with $4n+2-y, y-1, 4n+1-y$, and y. In Figure 2.5(c), before restoring the removed leg, four edge labels in the removed leg are $|x-y|$, $|4n+1-2y|$, $|4n+2-2y|$, and $|4n+3-2y|$. After restoring the removed leg, the new four edge labels in the removed leg are $|x-y|, |4n+3-2y|, |4n+2-2y|$, and $|4n+1-2y|$, i.e. T still has a graceful labelling, where the first vertex label of the body is 0.

Case 2: Assume that $T \notin \mathbb{T}_4$. Step 2 of algorithm A makes $T - u_1, u_2, ..., u_{j-1}$ be a new $T' \in \mathbb{T}_4$. By case 1, T' has a graceful labelling and the first vertex label of the body is 0. Labelling vertices $u_{j-1}, u_{j-2}, ..., u_1$ with according to the order $4n+1, -1, 4n+2$. Finally, we replace each vertex label s with $s+min u_1, u_2, ..., u_{j-1}$. In such construction, there yields a graceful labelling of T. Hence we complete the proof of the Theorem. □

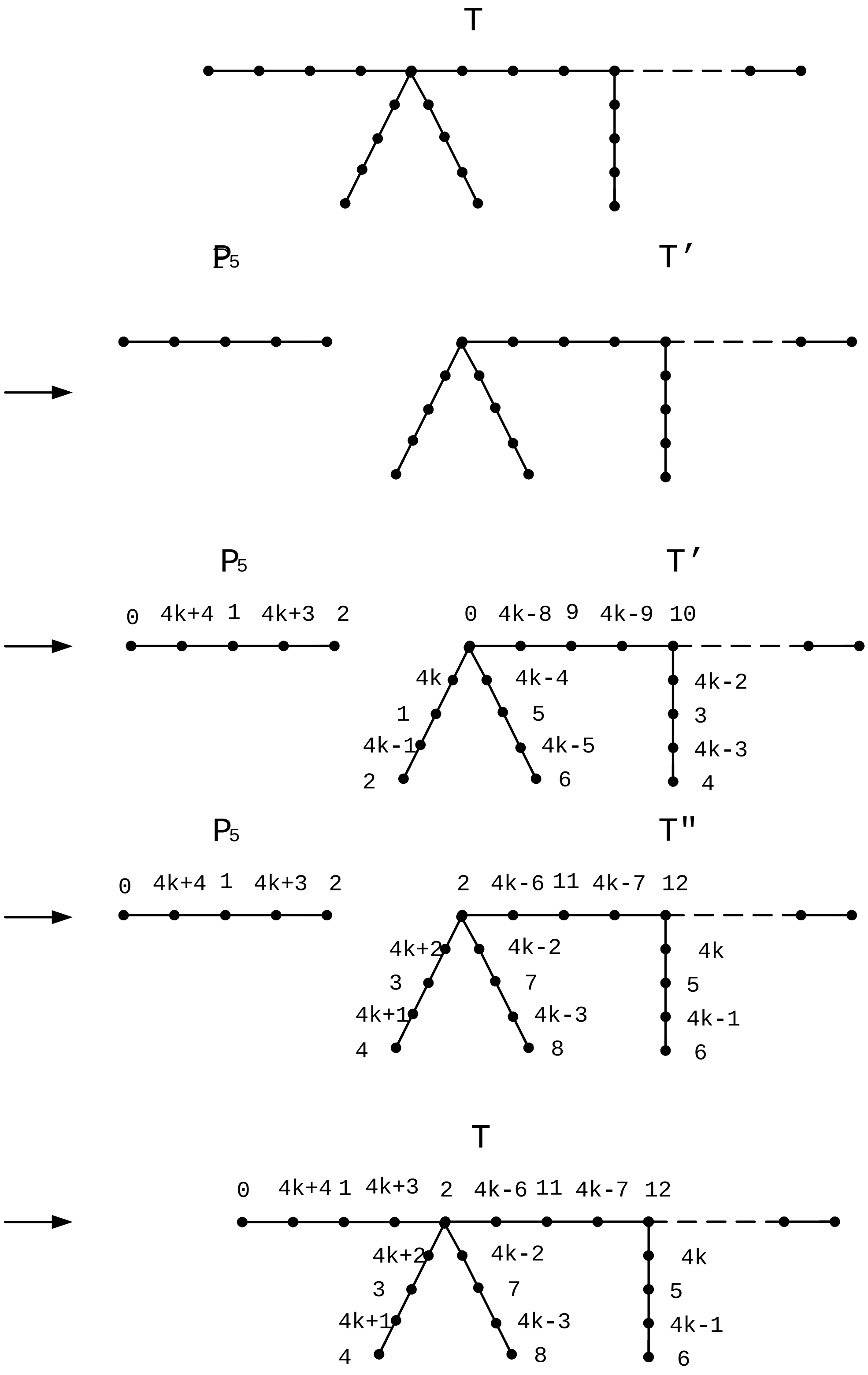

Figure 2.3: An example of the first vertex of the body in T has no legs.

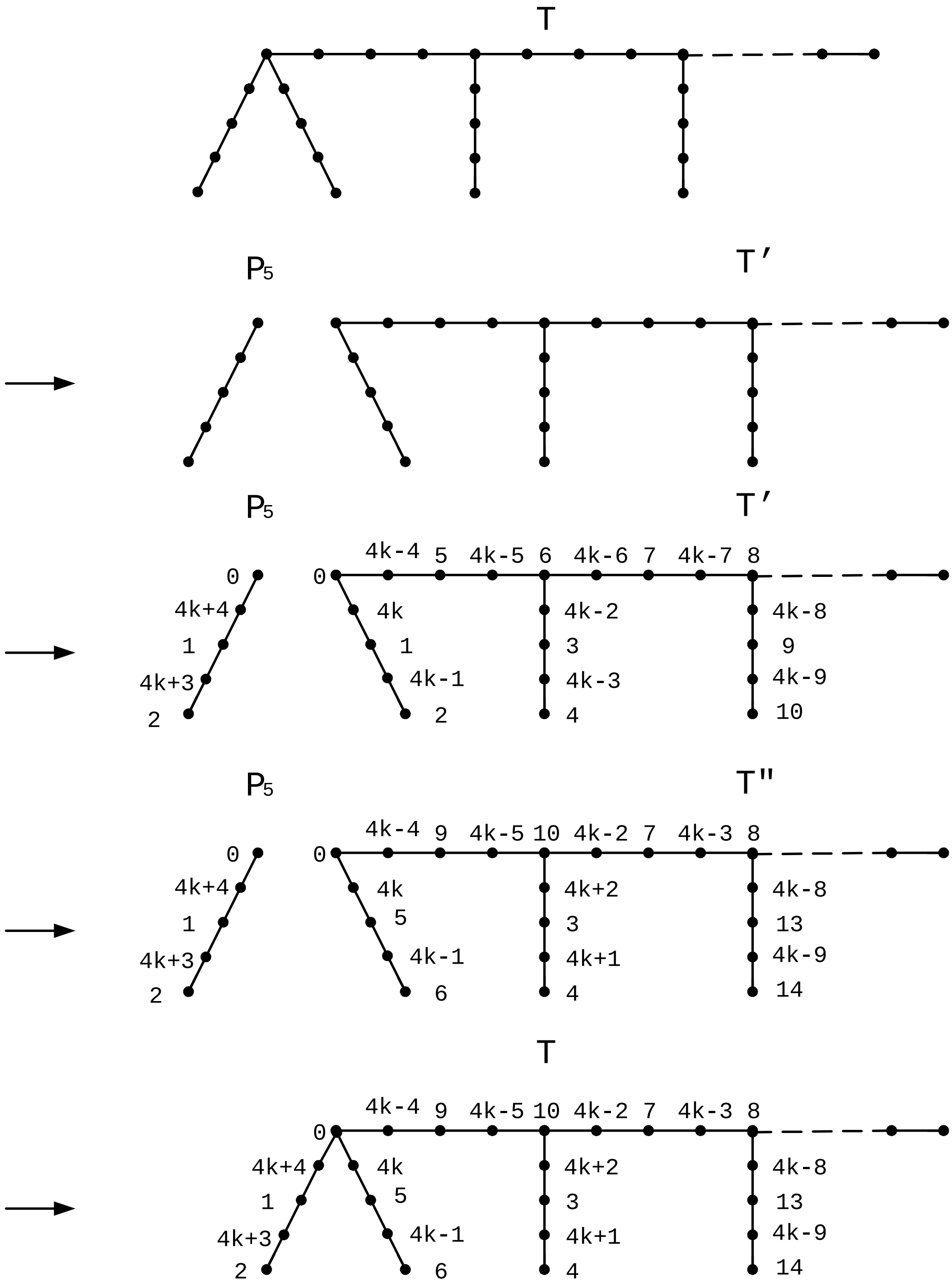

Figure 2.4: An example of the first vertex of the body in T has at least one leg.

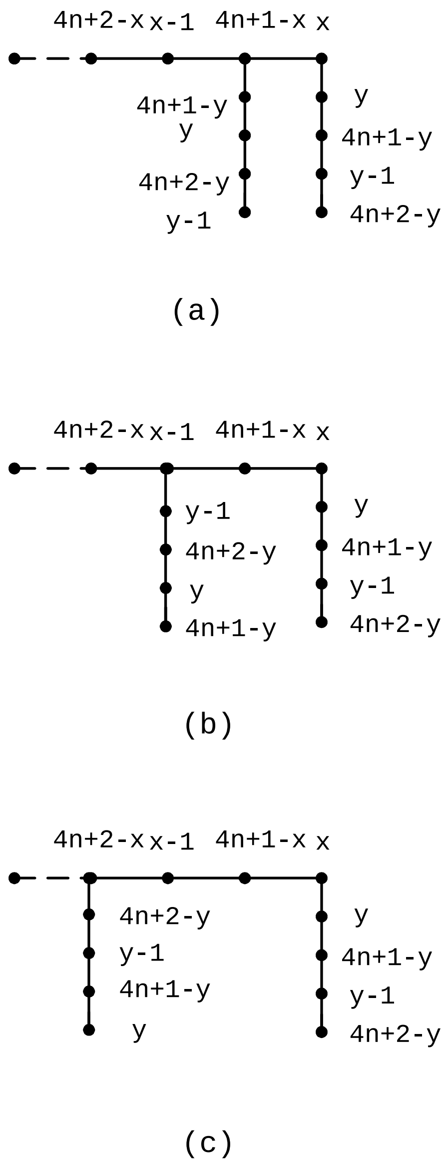

Figure 2.5: An illustration of theorem 2.02.