Incorporating Foreign Equities in the Optimal Asset Allocation of an Insurer with the Consideration for Background Risks: Models and Numerical Illustrations

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Abstract

We analyzes the optimal asset allocation problem for life insurers that are required to cope with significant background risks originating from life insurance businesses among a set of stochastic investment opportunities including foreign equities. The insurer is assumed to maximize the expected discounted utility of its surplus over a time horizon, and the optimalization problem is formulated in a stochastic control framework. Our derivations show that the optimal portfolio can be characterized by three components: a risk-minimizing component, a risk aversion index coupled with the portfolio's performance index, and the component reflecting the diffusions of state variables. Since the explicit solutions cannot be derived due to the complexity of the model, we employ Markov chain approximation methods to obtain the optimal control solutions numerically. The model and numerical methods are then applied to a hypothetical insurer in a simplified setting as an illustrative example.

Keywords: foreign investment; background risk; asset allocation; stochastic control; Markov chain approximation; life insurance

I. Introduction

The portfolio theory was developed originally to analyze the investment problem within a time invariant stochastic investment opportunity set within a single period. Markowitz (1952) shows how an investor chooses the assets with the forecasts about the assets' returns and risks in a single-period framework. The investor who cares only about the reward and risk of his/her portfolio will hold a specific portfolio of risky assets, i.e., the unique best mix of stocks and bonds. Tobin (1958) further derives the well-known separation theorem: the proportions of risky assets in the optimal portfolio are independent of the investor's risk aversion attitude. Many financial advisors, however, seem not to follow the mutual fund separation theorem. These advisors recommend more risk-averse investors to hold a higher ratio of bonds to stocks, which poses an asset allocation puzzle (Canner et al., 1997). A considerable number of studies have been made to solve this puzzle over the past several

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decades. For instance, Brennan and Xia (2000) show that the bond-stock mix is increasing with the risk-aversion parameter for the arbitrage beliefs about the risk premium vector.

The above single-period models are later extended to the multi-period framework in which the investment opportunities can be constant, time-dependent, or even stochastic. Merton (1971, 1973) explores the optimization of the dynamic portfolio in a multi-period framework given that the investment opportunity sets do not vary over time. Brennan, Schwartz, and Lagnado (1997) argue that Merton's analyses are logically flawed without considering the possibility of future changes in the investment opportunities. They consider the strategic asset allocation for long-term investors with time variations in expected returns of asset classes. Brennan (1998) later assumes that the investor could learn about the mean returns and demonstrates the impact of the learning on portfolio components. These papers show that the results of dynamic portfolio analyses differ significantly from those of single-period analyses and are greatly affected by the investment time horizon.

The trend of globalization and the rising importance of international financial markets inspire an extension of the portfolio theory in considering foreign investments. Popular foreign investments include stocks, bonds, real estate, mutual funds, and pooled trusts. Foreign investments are not just diversification components to domestic portfolios; they might help to mitigate interest rate risk. Campbell, Viceira, and White (2003) argue that domestic fixed income securities are risky for long-term investors because real interest rates vary over time and the investments need to be rolled over with uncertain future interest rates. They illustrate that the interest rate risk can be hedged by holding foreign currency if the domestic currency tends to depreciate when the domestic real interest rate falls.

In recent years, researchers broadened their attention from the risks in the financial markets to the risks outside the financial markets that are referred to as background risks. Background variables can be the investor's wage process, the contributions to and withdrawals from a pension fund, and the indemnities paid by the insurer to the insured. Menoncin (2002) models the background risks as a set of stochastic variables in analyzing the portfolio problem. By inserting the inflation risk that affects the growth rate of an investor's wealth, Menoncin derive an exact solution to the optimal portfolio problem when the financial market is complete. Menoncin also suggests an approximated general solution if the market is incomplete.

We in this paper incorporate foreign equities into the optimal asset allocation of a life insurer that faces significant background risks. The background risks of life insurance companies originate from the life/health insurance business books that involve mortality risk, morbidity risk, and/or surrender risk that are risks outside the financial markets. Since the life insurer has a high reserve-to-asset ratio, the consideration for the background risks is essential to find the insurer's optimal asset allocation strategy over time. The consideration for foreign investments is also indispensable because life insurers in many countries have significant positions in foreign investments. Furthermore, life insurers can use foreign investments to manage the interest rate risk embedded in the local financial markets, as Campbell, Viceira, and White (2003) suggest. Including background risks and foreign equities makes our model more realistic and comprehensive for insurance operations.

We model background risks and financial risks by two sets of state variables through stochastic processes. The investment opportunity set includes foreign equities. The goal of the insurer is to maximize the expected discount utility of surplus over a planning horizon. In order to improve investment performance over the planned horizon, the insurer needs to have a dynamic strategy to rebalance its portfolio consistently among various asset categories. We use the dynamic programming technique to tackle the optimization problem.

The Markov chain approximation method proposed in Kushner (1990) and Kushner and Dupuis (1992) is employed in demonstrating the numerical computations.[1]

Our derivations show that the optimal portfolio can be decomposed into three components: a hedging component minimizing the variance of changes in surplus, a component representing a familiar investment rule, and a component reflecting the diffusion matrix of state variables including background risks. Our numerical results show that the optimal asset allocation would vary with time and the insurer should adopt dynamic investment strategies rather than static ones. The optimal asset allocation also depends on the insurer's surplus level. Insurers therefore should change their investment strategies with their surplus levels and insurers of different sizes should adopt different strategies.

The rest of the paper is organized as follows. Section 2 details the stochastic differential equations that model the asset prices in the financial markets, the background variables, and the dynamic process of an insurer's surplus. Section 3 computes and discusses the compositions of the optimal portfolio. Section 4 presents the Markov chain approximation method that serves as the numerical method for the stochastic control problem in this study. Section 5 contains the application of our model and method to a hypothetical insurer. Section 6 concludes this paper.

II. The Model

Financial Markets. In this section, we set up a theoretical framework for the financial markets and specify the stochastic processes of the asset prices. We assume that the instantaneous spot rate follows the mean-reverting Ornstein-Uhlenbeck process proposed in Vasicek (1977):

$$dr(t) = q(m - r(t))dt + \vec{\sigma}_r(t)d\vec{W} \quad , \tag{1}$$

where:

m = the long-term average of spot rates ($m > 0$),

q = the speed of mean reverting ($q \in (0,1)$), $\vec{\sigma}_r(t) = [\sigma_r(t) \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$, and

$$d\vec{W} = [dW_r \quad dW_X \quad dW_S \quad dW_P \quad dW_{NCIF} \quad dW_L]' .$$

$d\vec{W}$ represents the differential of a six-dimension Wiener processes including the processes for the short rate (r), foreign equity index (X), domestic equity index (S), spot exchange rate (P), forecasting error of the net cash inflow from insurance liabilities ($NCIF$), and forecasting error in policy reserves (L). It has a correlation matrix Θ specifying the correlations among the Wiener processes. The dynamics of bond prices, according to Sorensen (1999), can then be described by:

$$\frac{dB(t,T)}{B(t,T)} = [r(t) + \lambda_r D(r,t)]dt - \vec{\sigma}_B d\vec{W} \quad , \tag{2}$$

[1] Recently, Monoyios (2004) applies this methodology to price European options with the presence of proportional transaction costs.

where:

$B(t,T)$ = the price of a bond with maturity $T-t$,

λ_r = the constant parameter describing the risk premium on interest rate risk,

$D(r,t) = -\dfrac{\frac{\partial B(t,T)}{B(t,T)}}{\partial r}$ is the elasticity of the bond price with respect to the short rate,

and

$$\vec{\sigma}_B = [\sigma_r D(r,t) \quad 0 \quad 0 \quad 0 \quad 0 \quad 0].$$

$D(r,t)$ is often referred to as the duration of an interest rate contingent claim.

The stock index is assumed to evolve according to the following process:

$$\frac{dS(t)}{S(t)} = (r(t) + \pi_S)dt + \vec{\sigma}_S d\vec{W}, \tag{3}$$

where the constant parameter π_S denotes the risk premium on the stock index investment and $\vec{\sigma}_S = [0 \quad 0 \quad \sigma_S \quad 0 \quad 0 \quad 0]$.

The investors can invest in a foreign equity portfolio. Let $P_f(t)$ and $X(t)$ denote the unit price of the foreign equity index and the spot exchange rate at time t, respectively. We refer to Björk (1998, chapter 12) in assuming the dynamics of $X(t)$ as:

$$\frac{dX(t)}{X(t)} = \alpha_X dt + \vec{\sigma}_X d\vec{W}, \tag{4}$$

where $\vec{\sigma}_X = [0 \quad \sigma_X \quad 0 \quad 0 \quad 0 \quad 0]$. The dynamics of the foreign equity index is assumed to have a similar form to the domestic equity index:

$$\frac{dP_f(t)}{P_f(t)} = (r_P(t) + \pi_P)dt + \vec{\sigma}_P d\vec{W}, \tag{5}$$

where r_P is the foreign riskless interest rate, π_P is a constant representing the expected excess return over r_P on the foreign equity portfolio, and $\vec{\sigma}_P = [0 \quad 0 \quad 0 \quad \sigma_P \quad 0 \quad 0]$.

Notice that one unit of the foreign stock portfolio is worth $P_f \times X$ in the domestic currency. Therefore, we have the following relation:

$$P_d(t) = P_f(t) \times X(t),$$

where P_d is the value of a hypothetical domestic asset portfolio that have an equivalent value to a unit foreign stock portfolio in terms of the domestic currency. Using the Itô's formula, we derive the dynamics of P_d as follows:

$$dP_d(t) = \frac{\partial P_d}{\partial t} dt + \frac{\partial P_d}{\partial P_f} dP_f + \frac{\partial P_d}{\partial X} dX + \frac{1}{2} \frac{\partial^2 P_d}{\partial P_f^2} (dP_f)^2 + \frac{1}{2} \frac{\partial^2 P_d}{\partial X^2} (dX)^2 + \frac{\partial^2 P_d}{\partial P_f \, \partial X} (dP_f)(dX).$$

Since $(\partial P_d / \partial t) = 0, (\partial^2 P_d / \partial P_f^2) = 0, (\partial^2 P_d / \partial X^2) = 0$, the price process P_d can be rewritten as:

$$dP_d(t) = P_d(t)[r_P(t) + \pi_P + \alpha_X + \sigma_P \sigma_X \rho_{PX}]dt + P_d(t)[0 \quad \sigma_X \quad 0 \quad \sigma_P \quad 0 \quad 0]d\bar{W} .$$

We summarize the state variables in the financial markets as follows:

$$\begin{aligned} dr(t) &= q(m - r(t))dt + \bar{\sigma}_r d\bar{W}, \\ dB(t,T)/B(t,T) &= [r(t) + \lambda_r D(r,t)]dt - \bar{\sigma}_B d\bar{W}, \\ dS(t)/S(t) &= (r(t) + \pi_S)dt + \bar{\sigma}_S d\bar{W}, \\ dP_f(t)/P_f(t) &= (r_P(t) + \pi_P)dt + \bar{\sigma}_P d\bar{W}, \text{ and} \\ dX(t)/X(t) &= \alpha_X dt + \bar{\sigma}_X d\bar{W}. \end{aligned}$$

The Investment Portfolio. Define $\vec{\theta}$ to be a (3 × 1) vector representing the proportions of risky assets:

$$\vec{\theta} = [\theta_1 \quad \theta_2 \quad \theta_3]',$$

where θ_1, θ_2, and θ_3 is the proportion of the wealth invested in bonds, domestic equities, and foreign equities respectively. The return process of the investment strategy $\vec{\theta}$ can then be written as:

$$R(t) = \theta_1 \frac{dB(t,T)}{B(t,T)} + \theta_2 \frac{dS(t)}{S(t)} + \theta_3 \frac{dP_d(t)}{P_d(t)} + (1 - \theta_1 - \theta_2 - \theta_3)r(t)dt . \tag{6}$$

Substituting the stochastic processes of individual asset returns into equation (6), we obtain:

$$R(t) = \mu_R dt + \bar{\sigma}_R d\vec{W} = [r(t) + \vec{\theta}' M]dt + \vec{\theta}' \Sigma d\vec{W} , \tag{7}$$

where:

$$M = \begin{bmatrix} \lambda_r D(r,t) \\ \pi_S \\ (r_P(t) - r(t)) + \pi_P + \alpha_X + \sigma_P \sigma_X \rho_{PX} \end{bmatrix} \text{ and}$$

$$\Sigma = \begin{bmatrix} -\sigma_r D(r,t) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_S & 0 & 0 & 0 \\ 0 & \sigma_X & 0 & \sigma_P & 0 & 0 \end{bmatrix}.$$

M represents the risk premiums of the risky assets and Σ contains the diffusion coefficients of the risky assets.

The Background Risks. In this paper, the variables representing background risks are the projection errors in the net cash inflows (gross premiums minus claims and expenses) and the errors in estimating liabilities associated with selling insurance during the investment time horizon.

Net Cash Inflows of Insurance Policies. The insurer selling multiple-premium products would experience cash outflows as well as inflows. The premiums from policyholders are cash inflows; underwriting expenses, claim payments, and claim adjustment expenses are cash outflows. The net cash inflows equal to the differences between the inflows and outflows are important to the insurer in terms of liquidity and growth. Insurers usually project future cash flows based on mortality, surrender, and management experiences. Since the projection is often done using conventional actuarial methods, the projected net cash inflow (*NCIF(t)*) is a deterministic function in *t*. On the other hand, projection errors are inevitable. We assume the projection error as follows:

$$dNCIF_{error}(t) = NCIF(t) \times \vec{\sigma}_{NCIF} \times d\vec{W} ,$$

where $\vec{\sigma}_{NCIF} = [0 \quad 0 \quad 0 \quad 0 \quad \sigma_{NCIF} \quad 0]$.

The Liabilities of Insurance Policies. Whenever an insurer collects premiums from policyholders, it is obliged to indemnify some or all losses incurred by the policyholders in the future. The value of the obligation estimated by actuaries (*L(t)*) is usually expressed as a deterministic function of *t*. Since estimation always couples with estimation errors, we specify that the estimation error obeys the following dynamics:

$$dL_{error}(t) = L(t) \times \vec{\sigma}_L \times d\vec{W} ,$$

where $\vec{\sigma}_L = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \sigma_L]$ describes the volatility of the error in estimating the liabilities.

The Insurer's Surplus. An insurer's surplus (*N*) is the excess of its asset values (*A*) over its liability values. Therefore, changes in an insurer's surplus equal to changes in its asset values minus changes in the liability values:

$$dN(t) = dA(t) - dL(t) - dL_{error}(t) .$$

The changes in asset values equal to the asset values at the beginning of a period multiply by the investment returns in the period plus the net cash inflows from insurance policies during the period. The relation can be expressed as follows:

$$dA(t) = A(t)R(t) + NCIF(t)dt + NCIF(t)\vec{\sigma}_{NCIF} d\vec{W}$$

$$= \left[A(r + \vec{\theta}' M) + NCIF\right]dt + \left(A\vec{\theta}'\Sigma + NCIF\vec{\sigma}_{NCIF}\right)d\vec{W} .$$

Changes in surplus can then be expressed as:

$$dN(t) = \left[(L+N)(r+\vec{\theta}'M) + NCIF - L\right]dt + \left[(L+N)\vec{\theta}'\Sigma + NCIF\vec{\sigma}_{NCIF} - L\vec{\sigma}_L\right]d\vec{W}\ . \qquad (8)$$

Equation (8) contains the state variables in the financial markets and the background variables outside the markets and will be fully explored in the subsequent analyses. Our goal is to obtain the optimal portfolio for insurers under the considerations for foreign equity investments and the background risks from selling life insurance.

III. The Optimal Portfolio

We assume that the insurer's objective is to maximize an expected discounted utility on its surplus over the time horizon [*0, T*]. Denote the utility function by $U(N)$ and assume that *U* is an increasing and concave function on the range of feasible values for *N*. More specifically, we assume that *U* is a Constant Relative Risk Aversion (*CRRA*) utility function so that the utility of instantaneous surplus is:

$$U(N(t)) = \begin{array}{l} \dfrac{N^{1-\gamma}}{1-\gamma} \text{ if } \gamma > 0 \text{ and } \gamma \neq 1 \\ log(N) \text{ if } \gamma = 1 \end{array}.$$

γ is set to be 0.5 to reflect the moderate risk preference of the insurer. The optimization problem for the insurer can then be formulated as:

$$\max_{\theta} E\left[\int_0^T e^{-\beta s} U\left(N\ ;\vec{\theta}\right)ds\right], \qquad (9)$$

given the dynamics:

$$d\begin{bmatrix} N \\ \vec{z} \end{bmatrix} = \begin{bmatrix} (L+N)(r+\vec{\theta}'M) + NCIF - L \\ \vec{\mu}_Z \end{bmatrix}dt + \begin{bmatrix} (L+N)\vec{\theta}'\Sigma + NCIF\vec{\sigma}_{NCIF} - L\vec{\sigma}_L \\ \Omega \end{bmatrix}d\vec{W}, \qquad (10)$$

where:

$$\vec{z} = \left[r\ X\ B\ S\ P_f\ NCIF_{error}\ L_{error}\right]',$$

$$\vec{\mu}_Z = \left[q(m-r)\quad X\alpha_X\quad B(r+\lambda_r D)\quad S(r+\pi_S)\quad P_f(r_P+\pi_P)\quad 0\quad 0\right]',$$

$$\Omega = \begin{bmatrix} \sigma_r & 0 & -B\sigma_r D & 0 & 0 & 0 & 0 \\ 0 & X\sigma_X & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & S\sigma_S & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & P_f\sigma_P & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & NCIF\sigma_{NCIF} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & L\sigma_L \end{bmatrix}', \text{ and}$$

$$\vec{z}(t_0) = \vec{z}_0,\ N(t_0) = N_0 \quad \forall t_0 \le t \le T .$$

We further impose a non-negative surplus constraint, $N(t) \ge 0$.

Given the above dynamic processes of the state variables and the non-negative capital requirement, the insurer seeks for the controller $\vec{\theta}$ over time to maximize the expected discounted utility of surplus. The conditional expectation given the information up to time *t* is defined as:

$$I(t,\vec{z},N;\vec{\theta}) = E_t\left[\int_t^T e^{-\beta s} U(N_s;\vec{\theta})ds\right].$$

Denote the optimal trading strategy $\vec{\theta}$ that maximizes $I(t,\vec{z},N;\vec{\theta})$ as $\vec{\theta} = arg\ max\, I(t,\vec{z},N;\vec{\theta})$ and the optimalized value function as $J(t,\vec{z},N;\vec{\theta}) = \sup_{\vec{\theta}} E_t\left[\int_t^T e^{-\beta s} U(N_s;\vec{\theta})ds\right]$, respectively. Then, the Bellman principal implies:

$$0 = \sup_{\vec{\theta}}\{E[dJ(t,\vec{z},N)]\}. \tag{11}$$

Using Itô's lemma, we obtain:

$$dJ(t,\vec{z},N) = J_t dt + J_N dN + J_{\vec{z}} d\vec{z} + \frac{1}{2} J_{NN}(dN)^2 + \frac{1}{2} J_{\vec{z}\vec{z}}(d\vec{z})^2 + J_{N\vec{z}} dN d\vec{z},$$

where the subscripts denote partial derivatives. The differential equation of the optimal value function can be computed using the dynamics of *dN* and $d\vec{z}$ in equation (10). We then substitute *dJ* into equation (11) to obtain an equation that depends on both $\vec{\theta}$ and the optimal value function as follows:

$$\begin{aligned} 0 = J_t &+ \sup_{\vec{\theta}}\{ J_N\left[(L+N)(r+\vec{\theta}' M) + NCIF - L\right] + J_{\vec{z}}\vec{\mu}_{\vec{z}} \\ &+ \frac{1}{2} J_{NN}\left[(L+N)\vec{\theta}'\Sigma + NCIF\vec{\sigma}_{NCIF} + L\vec{\sigma}_L\right]^2 + \frac{1}{2} J_{\vec{z}\vec{z}}\Omega'\Omega . \\ &+ J_{N\vec{z}}\left[(L+N)\vec{\theta}'\Sigma + NCIF\vec{\sigma}_{NCIF} - L\vec{\sigma}_L\right]\Omega' \},\ 0 \le t \le T \end{aligned} \tag{12}$$

Equation (12) is known as the Hamilton-Jacobi-Bellman (*HJB*) equation. In other words, our optimal control problem is equivalent to the problem of finding a solution of the *HJB* equation.

Taking derivatives for equation (12) with respect to $\vec{\theta}$ and using the first order condition, we can present the optimal asset allocation vector $\vec{\theta}^*$ as:

$$\begin{aligned}\vec{\theta}^{*} = & \frac{L}{(L+N)}(\Sigma'\Sigma)^{-1}\Sigma\vec{\sigma}_{L}' - \frac{NCIF}{(L+N)}(\Sigma'\Sigma)^{-1}\Sigma\vec{\sigma}_{NCIF}' \\ & - \frac{J_N}{J_{NN}}\frac{1}{(L+N)}(\Sigma'\Sigma)^{-1}M \\ & - \frac{J_{N\vec{z}}}{J_{NN}}\frac{1}{(L+N)}(\Sigma'\Sigma)^{-1}\Sigma\Omega'\end{aligned} \quad . \qquad (13)$$

Equation (13) represents the optimal allocation among asset classes and can be decomposed into three components:

1. $\frac{L}{(L+N)}(\Sigma'\Sigma)^{-1}\Sigma\vec{\sigma}_{L}' - \frac{NCIF}{(L+N)}(\Sigma'\Sigma)^{-1}\Sigma\vec{\sigma}_{NCIF}'$. This component is unrelated to the investor's preference, but depends primarily on the diffusion matrix of the risky assets (Σ), the diffusion term of the background risks ($\vec{\sigma}_L$, $\vec{\sigma}_{NCIF}$), and the debt ratio $\frac{L}{L+N}$.

2. $-\frac{J_N}{J_{NN}}\frac{1}{(L+N)}(\Sigma'\Sigma)^{-1}M$. This component is the product of the inverse of Arrow-Pratt risk aversion index $-\frac{J_N}{J_{NN}}$ and the portfolio Sharpe ratio ($(\Sigma'\Sigma)^{-1}M$) normalized by the size $\frac{1}{L+N}$.[2]

3. $-\frac{J_{Nz}}{J_{NN}}\frac{1}{(L+N)}(\Sigma'\Sigma)^{-1}\Sigma\Omega'$. In addition to the partial derivatives of the value function, size, and the diffusion terms of the risky assets, this component also depends on the diffusion matrix of the state variables (Ω).

The first component minimizes the instantaneous variance of the surplus differential since the variance of the changes in surplus (*dN*) in equation (8) is:

$$\begin{aligned}Var(dN) = & (L+N)^2\vec{\theta}'\Sigma\Sigma'\vec{\theta} + CF^2\vec{\sigma}_{CF}\vec{\sigma}_{CF}' + L^2\vec{\sigma}_L\vec{\sigma}_L' \\ & + 2(L+N)CF\vec{\theta}'\Sigma\vec{\sigma}_{CF}' - 2(L+N)L\vec{\theta}'\Sigma\vec{\sigma}_L' - 2CFL\vec{\sigma}_{CF}\vec{\sigma}_L'.\end{aligned}$$

The second component is a familiar portfolio rule. It indicates that the optimal asset allocation to risky assets is inversely related to the Arrow-Pratt risk aversion index and is proportional to the portfolio's Sharpe ratio. The third component reflects how the state variables in the financial and insurance markets affect the optimal investment portfolio.

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[2] The Arrow-Pratt risk aversion index represents the investor's preference. An investor is risk-averse when the Arrow-Pratt risk-aversion index is greater than zero. The portfolio Sharpe ratio is the ratio of the assets’ excess returns to the returns’ variances.

IV. The Numerical Solution

In the previous sections, we formulate the optimal investment strategy of an insurer as a stochastic control problem in a continuous-time framework. Solving the *HJB* equation analytically, however, is difficult. The *HJB* equation has been solved in some particular cases only and the non-negative surplus constraint makes the task more difficult. Even if the solution can be obtained, the required long-winded technicalities are awkward for practical uses. Therefore, we employ the Markov chain approximation method proposed by Kushner and Dupuis (1992) to solve the control problems numerically.

The basic idea behind the Markov chain approximation is to approximate the original controlled problem by a simpler one for which the computation is workable, and then prove that the approximation converges to the original problem when the parameter goes to its limit. The proofs for the convergence of the Markov chain approximation are purely probabilistic. Since the details and generalizations have been discussed by Kushner and Dupuis (1992), we illustrate only the expression form of the Markov chain approximation method in this section and explain briefly the approximation procedures.

The approximating chains can be loosely divided into explicit and implicit methods according to the treatment of time. In the explicit method, the time variable is a true "time" variable that increases by a constant at each step. In the implicit one, the time variable is treated as another state variable. The approximating Markov chain has a state space that is a discretization of the t (time state variable) space and the x (other state variables) space, and the time variable does not necessarily increase its value at each step. We adopt the implicit approximation method because of its superior converging speed and numerical accuracy.

The Expression from the Markov Chain Approximation Method. We consider a general finite difference method with one dimension control process in the following. Assume that a controlled process x satisfies the stochastic differential equation:

$$dx = b(x,\theta)dt + \sigma(x,\theta)dW \ ,$$

where θ denotes the controller and dW is the differential of the Wiener process. We can regard the control problem as:

$$I(x,t,\theta) = E^{\theta}_{x,t}\left[\int_t^T e^{-\beta s} k(x,s,\theta) ds + g(x(T),T)\right],$$

and define the optimal value function as:

$$J(x,t) = \max_{\theta} I(x,t,\theta) \, .$$

Then, the formal dynamic programming equation for the optimal value function is:

$$\begin{aligned} 0 &= J_t(x,t) + \max_{\theta}\left[A^{\theta} J(x,t) + k(x,t)\right] \\ &= J_t(x,t) + \max_{\theta}\left[\frac{1}{2}\sigma^2(x,\theta) J_{xx}(x) + b(x,\theta) J_x(x) + k(x,\theta)\right], \end{aligned}$$

where A^{θ} denotes the differential operator of the controlled process x with respect to the controller θ.

Let $p^{h,\delta}(x,y\,|\,\theta)$ and $\Delta t^{h,\delta}$ be the transition probability of the implicit method and the interpolation interval that are locally consistent with the controlled diffusion process, respectively. Referring to Kushner and Dupuis (1992), we formulate the dynamic programming equation for the approximation optimal control problem as:

$$J^{h,\delta}(x,n\delta)\Big|_{\theta_i} = \max_{\theta} \sum_{y} p^{h,\delta}(x,n\delta;y,n\delta) J^{h,\delta}(y,n\delta)\Big|_{\theta_i} + e^{-\beta\delta} p^{h,\delta}(x,n\delta;x,n\delta+\delta|\theta) J^{h,\delta}(x,n\delta+\delta)\Big|_{\theta_i} + k(x,\theta)\Delta t^{h,\delta}$$

and

$$\theta_{i+1} = \arg\max_{\theta_i}\{\sum_{y} p^{h,\delta}(x,n\delta;y,n\delta) J^{h,\delta}(y,n\delta)\Big|_{\theta_i} + e^{-\beta\delta} p^{h,\delta}(x,n\delta;x,n\delta+\delta|\theta) J^{h,\delta}(x,n\delta+\delta)\Big|_{\theta_i} + k(x,\theta)\Delta t^{h,\delta}\},$$

where:

h and *δ* are the space increment and time increment respectively,

$$\Delta t^{h,\delta} = \frac{\delta h^2}{h^2 + \overline{Q}^{h,\delta}\delta}, \quad p^{h,\delta}(x,n\delta;x,n\delta+\delta|\theta) = \frac{h^2}{h^2 + \overline{Q}^{h,\delta}\delta},$$

$$p^{h,\delta}(x,n\delta;y,n\delta) = \frac{V(x,y,\theta)\delta}{h^2 + \overline{Q}p^{h,\delta}\delta}, y \neq x,$$

$$p^{h,\delta}(x,n\delta;x,n\delta) = \frac{\delta(\overline{Q}^{h,\delta} - Q^{h,\delta}(x,\theta))}{h^2 + \overline{Q}^{h,\delta}\delta},$$

$$\delta = \frac{h^2}{\overline{Q}^{h,\delta}}, \text{ and}$$

$$V(x,x\pm h,\theta) = \frac{\sigma^2(x,\theta)}{2} + \max(\pm b(x,\theta),0)h$$
$$= 0 \text{ if } y \neq x \pm h.$$

The Approximation Procedure. The procedures of the approximation scheme are explained as follows. First, we select an initial controller θ_0 and compute the transition probabilities and interpolation interval respectively. Then the corresponding value functions of the given controller at each *x* and the terminal time *T* are solved, and we work backwardly to obtain all value functions at each time stage. Finally, we select the maximizer (a new controller) θ_1 to maximize the value function given by all obtained value functions for each *x* and *t*, and repeat the iteration to reach the final optimal value function and the optimal controller. A Genetic Algorithm approach is used in the final step to obtain the optimal value function and the optimal control.

Table 1: Compositions of Liability Portfolio of a Hypothetical Insurer

Product	Face Amount (NT dollar)	Number of Policies	Premium Income (NT dollar)	Policy Reserves (NT dollar)
Whole Life	$1 million	426,251	$6.29 billion	$69.58 billion
Term Life	$1.5 million	12,501	$0.09 billion	$0.11 billion
Endowment	$1 million	93,774	$4.51 billion	$24.29 billion
Endowment with Bi-Year Survival Benefits	$0.5 million	648,606	$6.12 billion	$58.00 billion
Total		1,181,132	$17.01 billion	$151.98 billion

Table 2: Decomposition of Liability Portfolio by Premium Payment Periods/Policy Maturities

	Single Premium		Level Premium				
Policy Maturity and/or Premium Payment Period	6	10	6	10	15	20	Total
Whole Life	2.0%			5.0%	10.0%	20.0%	37.0%
Term Life			0.1%	0.1%	0.1%	0.2%	0.5%
Endowment	10.0%	5.0%	8.0%	2.5%	0.5%	0.5%	26.5%
Endowment with Bi-Year Survival Benefits			7.0%	12.0%	7.0%	10.0%	36.0%
Total	12.0%	5.0%	15.1%	19.6%	17.6%	30.7%	100.0%

Notice that our controlled process has a complex form:

$$dN = \mu_N dt + \vec{\sigma}_N d\vec{W} \ ,$$

where:

$$\mu_N = (L+N)\left[r + \theta_1 \lambda_r D + \theta_2 \pi_S + \theta_3 (r_P + \pi_P - r + \alpha_X + \sigma_P \sigma_X \rho_{PX})\right] + NCIF - L \text{ , and}$$

$$\vec{\sigma}_N = \left[-(L+N)\theta_1 \sigma_r D \quad (L+N)\theta_3 \sigma_X \quad (L+N)\theta_2 \sigma_S \quad (L+N)\theta_3 \sigma_P \quad NCIF\sigma_{NCIF} \quad -L\sigma_L\right]$$

The dynamic programming equation for our control problem is therefore quite complicated. The features that $V(x, x \pm h, \theta)$ contains the maximum function and is a non-continuous function make it even more difficult to solve for the optimal value function and the optimal controller in the final procedures. Therefore, solving such a compound control problem with the Markov chain approximation method demands extensive computation power.

V. An Illustrative Example

The Specification of a Hypothetical Insurer's Businesses. We construct the liability portfolio of the hypothetical insurer based on a portion of businesses of a mid-size life insurer in Taiwan. For simplification, we assume that the insureds are all male with age of

35. The mortality rates for the insureds are assumed to be 54% of those in the 1989 Taiwan Standard Ordinary Tables of Mortality (89TSO).[3] The compositions of the portfolio are listed in Table 1.

They can be further decomposed by premium payment period, as seen in Table 2.[4]

Since these policies are sold over a period of several years, their pricing rate and premium rate differ from each other. The premium rates by types of insurance, premium payment period/policy maturity, and pricing rate are listed in Table 3.

Denote the portfolio of the type j policy (j=1~4) by Ψ_j and the premium income at time t from the insured i holding the type j policy by $PM_{ij}(t)$. Then, the cash inflow of annual premium income is $\sum_{j=1}^{4}\sum_{i\in\Psi_j} PM_{ij}(t)$. To project $PM_{ij}(t)$, we need not only mortality rates but also surrender rates. The surrender rates are assumed as follows.[5]

Policy Year	1	2	3	4	5	6	7 - 9	10 - 20	21 -40
Surrender Rate	20%	10%	9%	8%	7%	6%	5%	3%	1%

The outgoing expenses and claim payments at time t due to the ij-th policy are denoted by $E_{ij}(t)$ and $C_{ij}(t)$, respectively. Operational expenses are projected using the following table.

Policy Year		Single Premium	Level Premium
1st		8%*Premium + $700	10%*Premium + $700
2nd and later	Premium Payment Period	2.5%*Premium + $200	2.5%*Premium + $200
	Non-Premium Payment Period	2.5%*Premium + $100	2.5%*Premium + $100

The insurer has to pay 0.1% of premiums income to the guarantee fund in addition to the operational expenses.[6] Furthermore, these policies are all participating policies with

[3] 54% is the common assumption used among actuaries in Taiwan since the mortality table is constructed a decade ago and over-estimates the mortality rate.

[4] The prepayment payment period usually matches with the policy maturity, with the exceptions of whole life insurance and single-premium policies. The single-premium endowment policies have two maturities: six and ten years.

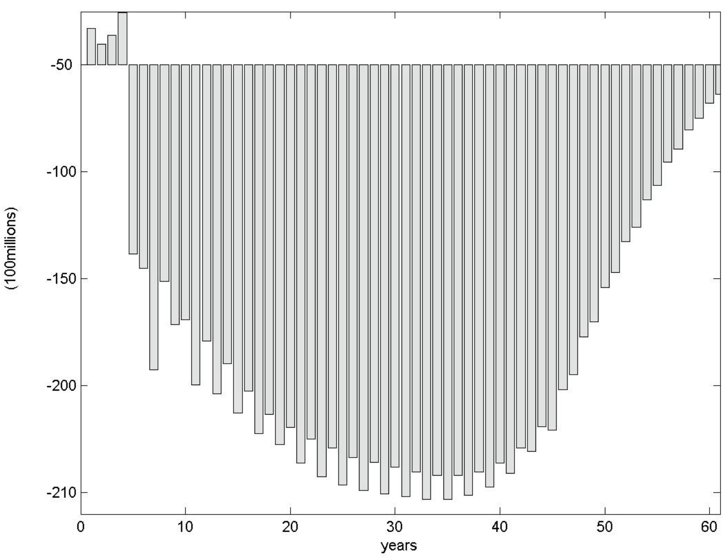
[5] For simplicity, we assume that the surrender behaviors of policyholders are the same across types of policy.

[6] We omit some detailed assumptions in projecting cash flows, e.g., commission rate and the further break-down of portfolio compositions by premium rate, for the sake of paper length.

Table 3: Premium Rates per $100,000 Face Amount

	Premium Payment Method	Policy Maturity and/or Premium Payment Period	Pricing Rate					
			7.50%	7.00%	6.75%	6.50%	6.25%	4.00%
Whole Life	Single			13,432	14,254	15,147	16,117	29,600
	Level	10 Year	1,814	1,998	2,101		2,212	3,973
		15 Year	1,420	1,552	1,625		1,704	2,922
		20 Year	1,347	1,461	1,525		1,593	2,624
Term Life	Level	6 Year		297	297	298	298	303
		10 Year		346	347	348	349	358
		15 Year		437	439	441	443	463
		20 Year		590	590	595	603	644
Endowment	Single	6 Year		80,549	81,677	82,823	83,989	95,392
		10 Year		61,939	63,375	64,848	66,359	81,877
	Level	6 Year		15,879	16,015	16,152	16,291	17,596
		10 Year		8,327	8,442	8,559	8,677	9,813
		15 Year		4,710	4,806	4,904	5,004	5,995
		20 Year		3,700	3,799	3,900	4,004	5,066
Endowment with Bi-Year Survival Benefits	Level	6 Year		2,649	2,796	2,955	3,127	5,529
		10 Year		1,947	2,047	2,156	2,272	3,873
		15 Year		1,512	1,583	1,659	1,742	2,846
		20 Year		1,310	1,367	1,428	1,493	2,353

Figure 1: Projected Net Cash Inflows



dividend payments determined by a regulatory formula.[7] The cash outflow associated with the insurance pool at time t can then be expressed as $\sum_{j=1}^{4}\sum_{i\in\Psi_j}(E_{ij}(t)+C_{ij}(t))$.

Finally, the projected net cash inflow from the insurance pool at time t equals:

$$NCIF(t)=\sum_{j=1}^{4}\sum_{i\in\Psi_j}(PM_{ij}(t)-E_{ij}(t)-C_{ij}(t)).$$

The projection results are shown in Figure 1.

We also need to project policy reserves at future times. Since policy reserves are the difference between the present value of future benefit payments and that of future net premiums, the projected policy reserves equal:

$$L(t)=\sum_{j=1}^{4}\sum_{i\in\Psi_j}(A_{n-t}(ij)-NP_{ij}\times\ddot{a}_{x_{ij}:\overline{n-t|},r\%}),$$

where NP_{ij} is the net premium from the ij-th policy, $A_{n-t}(ij)$ denotes the discount value of future benefit payments, and $\ddot{a}_{x_{ij}:\overline{n-t|},r\%}$ is the present value of an annuity due with n-t payments at attained age x_{ij} and discount rate of r%. We need loading rates to calculate NP_{ij} from gross premiums PM_{ij}. The loading rates are listed in Table 4 and the projected reserves are shown in Figure 2.

To meet its obligation, the insurer invests its surplus and the collected premiums into the financial markets. We estimate the parameters for the dynamic process of the domestic stock index using the empirical data in Taiwan and obtain the following result:

$$\frac{dS(t)}{S(t)}=0.1550dt+0.1380dW_S.$$

Using past S&P 500 index and exchange rates between U.S. and NT dollars, we obtain the following estimated dynamic processes:

$$\frac{dX(t)}{X(t)}=0.0532dt+0.0157dW_X \text{ and } \frac{dP_f(t)}{P_f(t)}=0.0353dt+0.0191dW_P.$$

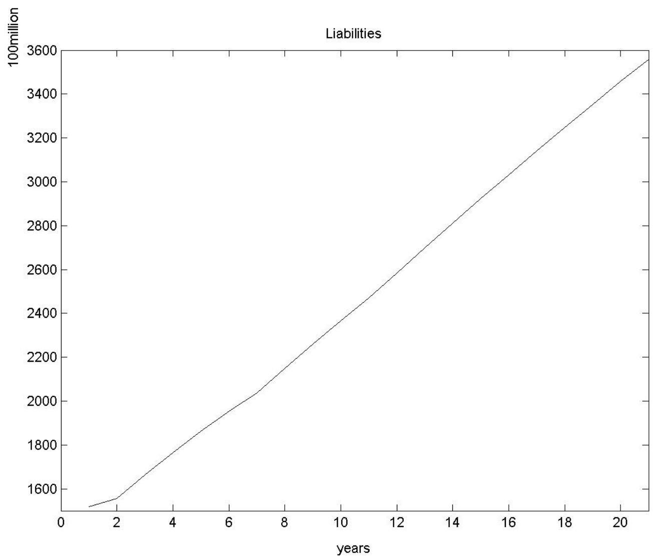
The correlations between the state variables are set to be $\rho_{XP}=-0.0612$ and $\rho_{SP}=-0.0257$. The initial values for $X(t)$, $S(t)$, and $P_f(t)$ are 35, 4,500, and 900.

[7] All life insurance polices sold before 2004 in Taiwan are required to pay dividend to policyholders according a formula set by the insurance supervisor. The mortality rate dividend is equal to the difference between the assumed mortality rates and 90% of those in the 89 TSO multiplied by the net amount at risk. The investment return dividend equal to the policy reserve times the difference between the pricing rate and a short-term interest rate.

Table 4: Loading Rates by Types of Policies and Premium Payment Periods

Premium Payment Period	Single Premium	6	10	15	20
Whole Life	17%		25%	25%	31%
Term Life		17%	23%	23%	32%
Endowment	17%	17%	23%	23%	23%
Endowment with Bi-Year Survival Benefits	17%	17%	23%	23%	23%

Figure 2: Projected Policy Reserves of the Hypothetical Insurer



The estimated processes for the short rate and bond return are:

$$dr(t) = 0.0564(0.0626 - r(t)) + 0.009dW_r \text{ and}$$

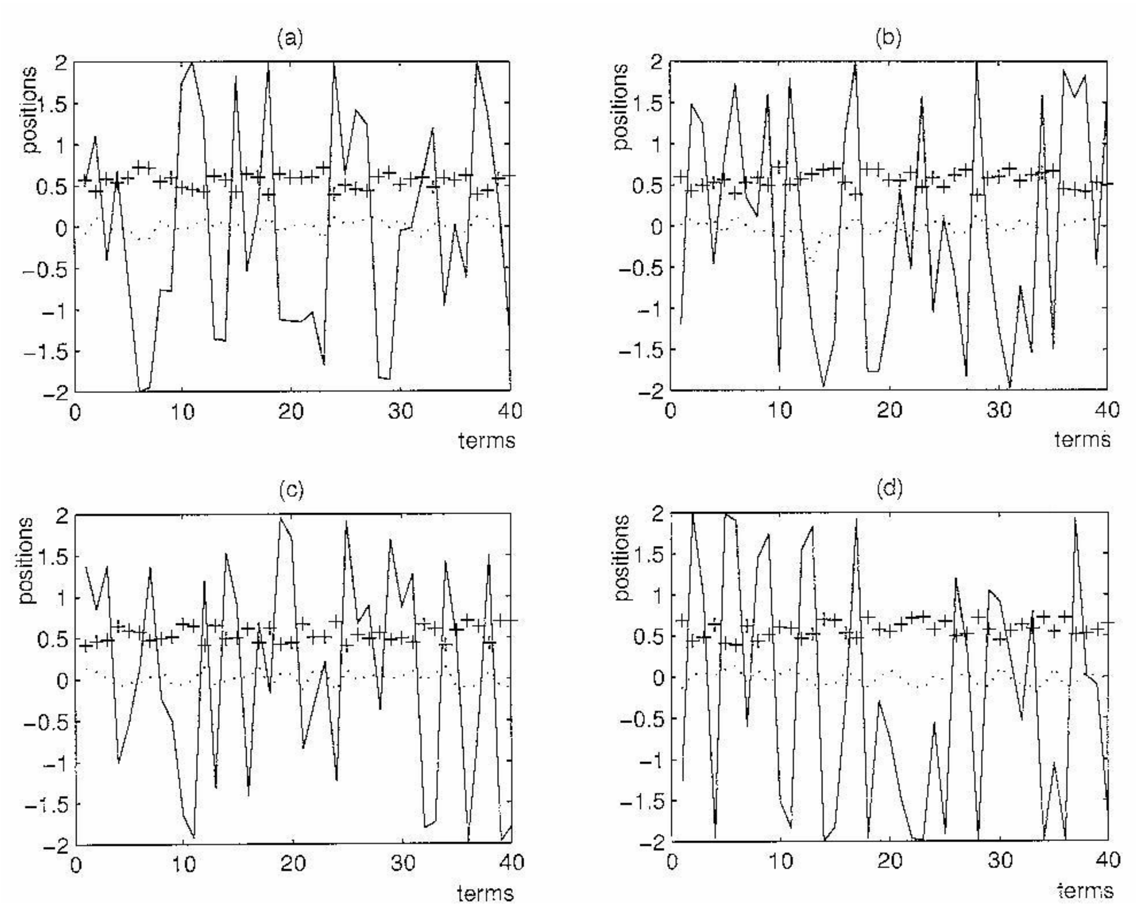
$$\frac{dB(t)}{B(t)} = \left[r(t) + 0.0281(1 - e^{-0.0564(T-t)})\right]dt - 0.1604(1 - e^{-0.0564(T-t)})dW_r.$$ [8]

The initial values for the short rate and bond price are 2% and 1,000, respectively.

Since the state variables are six-dimension vectors in the above model, it is impossible to incorporate the entire state vectors into the numerical calculation due to computational constraints. Therefore, only the price process of the foreign asset is modeled as a

[8] The parameters of the short rate process and bond return process are estimated using GMM (generalized method of moments) with monthly 30-day commercial paper rate from November 1980 to February 2001. The data source is TEJ (Taiwan Economic Journal) databank.

Figure 3: Optimal Asset Allocations to Bonds, Domestic Equities, and Foreign Equities over Time Given Four Levels of Surplus (---: bonds; +++: domestic stocks; …: foreign stocks)



stochastic process and the values of other state variables are determined through simulations.[9] Furthermore, we set the time horizon as 5 years instead of 20 years or 60 years for the reason of computation time. The time horizon is further divided into 40 grids/terms in the computation.

Numerical Results. Since the optimal asset proportion $\vec{\theta}^*$ is the function of surplus and time, we present two sets of figures and tables in which one variable is fixed while the other varies.[10] Figure 3 displays the optimal asset allocations to bonds, domestic equities, and foreign equities over time, given the surplus of (a) \$100 billion, (b) \$150 billion, (c) \$200

[9] Owing to the lack of information on the cash flow volatilities of the policies, we assume a variance of 2.25% in projecting the net cash inflows and reserves.

[10] The optimal asset allocation is of course a function of all parameters of the stochastic processes considered in the model as well. We do not perform comprehensive sensitivity analyses because of the computation time and the length of the paper.

Table 5: Means and Standard Deviations of the Allocations to Bonds, Domestic Equities, and Foreign Equities in Figure 3

Surplus Level	Allocation	Mean	Standard Deviation
(a) $100 billion	Bond	-0.1214	12.9181
	Domestic Equities	5.5270	0.9128
	Foreign Equities	0.0593	0.8166
(b) $150 billion	Bond	0.1318	13.8489
	Domestic Equities	5.6177	0.9405
	Foreign Equities	-0.2053	1.0820
(c) $200 billion	Bond	0.6790	12.8533
	Domestic Equities	5.4357	0.9718
	Foreign Equities	0.0819	0.8810
(d) $250 billion	Bond	-2.4430	14.8375
	Domestic Equities	5.7763	0.9692
	Foreign Equities	0.2266	0.8374

billion, and (d) $250 billion.[11] Table 5 exhibits the means and standard deviations of the allocations in Figure 3.

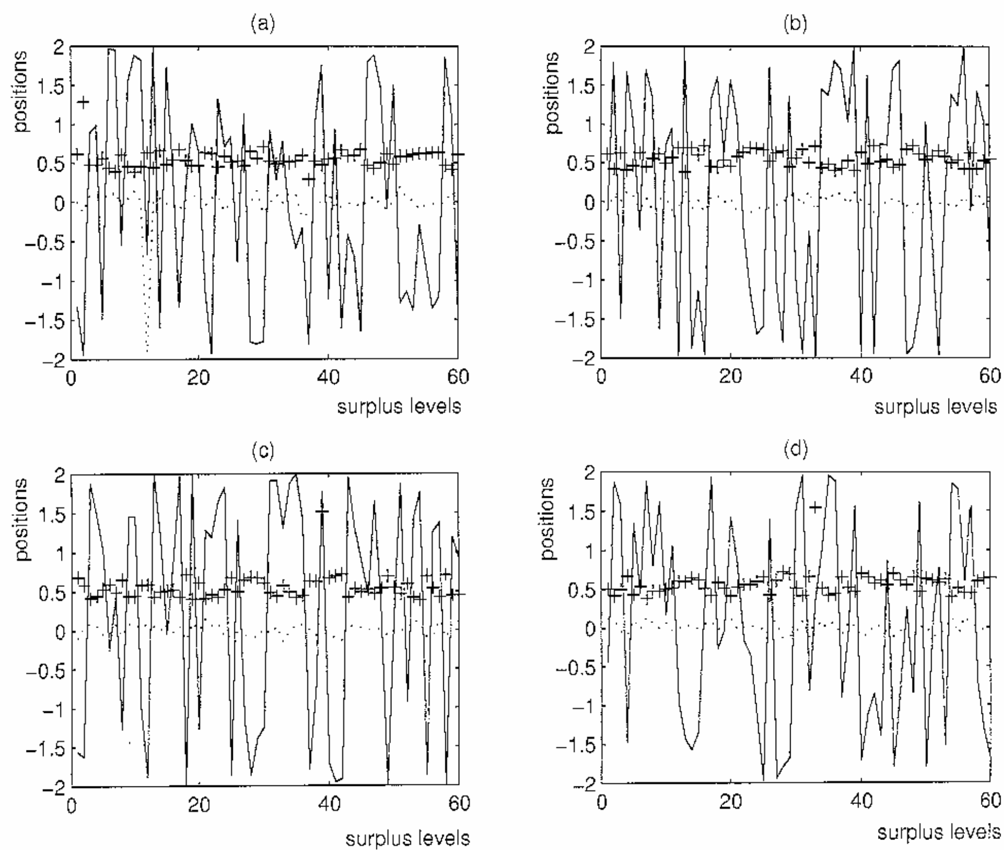
The apparent feature shown by Figure 3 and Table 5 is that the optimal asset allocation of the insurer varies with time to a significant extent. Static investment strategies therefore are not appropriate. Figure 3 and Table 5 further show that the insurer invests more in domestic equities than in foreign equities. The positions in these two types of equities have similar standard deviations while the position in bonds is rather volatile.

Figure 4 shows the optimal asset allocations to bonds, domestic equities, and foreign equities over surplus levels at the time of the 15^{th}, 30^{th}, 45^{th}, and 60^{th} month. The range of surplus level is assumed to be between 50 and 250 billion NT that is further divided into 60 levels as shown in Figure 4. Table 6 contains the means and standard deviations of the allocations in Figure 4. Figure 4 and Table 6 illustrate that the optimal asset allocations are different across surplus levels. In other words, the insurer should adopt different investment strategies according to its surplus level. Figure 4 and Table 6 also show that the insurer invests more in domestic equities than in foreign equities across the four points of time. The position in bonds is, again, rather volatile.

The above two sets of figures and tables suggest that the insurer should invest more in domestic equities than in foreign equities, which could be due to the higher expected return in the domestic stock market during the sampling period. We also find that the insurer holds

[11] The scale of the Y-axis has been divided by 10, i.e., the 1.5 in the figure is indeed 15. The same scaling applies to Figure 4 as well. The proportions allocated to bonds, domestic equities, and foreign equities do not add up to 1 because we do not report the proportions in cash.

Figure 4: Optimal Asset Allocations to Bonds, Domestic Equities, and Foreign Equities over Surplus Levels at Four Points of Time (---: bonds; +++: domestic stocks; …: foreign stocks).



higher-than-average positions in bonds when its positions in domestic equities are lower than average.[12] This implies that the insurer dynamically adjusts its allocations between domestic equities and bonds within the local markets while uses the foreign investment positions as an underlying diversification/hedging component.

VI. Conclusions

This paper studies the optimal investment problem of insurance companies that face not only financial risks but also the background risks from underwriting life insurance. Foreign equities are considered in the model to broaden the investment/hedging opportunities. We assume that the insurer's objective is to maximize its expected discounted utility of the surplus over a target investment horizon and formulate the problem in the stochastic control framework. After specifying the dynamic structure of the financial asset prices and projecting future cash flows associated with insurance liabilities, we derive the optimal asset allocation vector and use the Markov chain approximation method to seek numerical

[12] This finding cannot be observed with bare eyes from the figures. We find it through running simple correlation analyses.

Table 6: Means and Standard Deviations of the Allocations to Bonds, Domestic Equities, and Foreign Equities in Figure 4

Time	Allocation	Mean	Standard Deviation
15th Month	Bond	0.1271	13.3032
	Domestic Equities	5.4983	1.2916
	Foreign Equities	-0.1974	2.6223
30th Month	Bond	1.4570	14.8670
	Domestic Equities	5.5129	0.9967
	Foreign Equities	-0.0864	0.8403
45th Month	Bond	3.0489	14.8910
	Domestic Equities	5.5345	1.6039
	Foreign Equities	0.0565	0.7633
60th Month	Bond	0.4061	13.2319
	Domestic Equities	5.6575	1.5613
	Foreign Equities	0.0740	0.7744

solutions. Since the overall approximating procedures are quite time-consuming, the majority of the state variables are projected through simulations in this work.

The optimal asset allocation vector contains three components: (i) the preference-free component that minimizes the instantaneous variance of changes in the surplus; (ii) the familiar myopic portfolio rule that is chiefly concerned with the inverse of the Arrow-Pratt risk aversion index and the portfolio's Sharpe ratio; and (iii) the component dependent upon the diffusion matrix of all state variables including background risks.

Our numerical results demonstrate that the optimal asset allocation of the insurer varies with time to a significant extent, which justifies the use of the dynamic control method over conventional static one. The results also show that the optimal asset allocations are different across surplus levels. The insurer therefore should adopt different investment strategies according to its surplus level. We further find that the hypothetical insurer will invest more in the domestic stock market than in the foreign equity market, which is probably due to the higher expected return of the domestic stock market. In addition, we find that the insurer tends to shifts its funds to the bond market when the domestic stock market generates low returns while leaves the foreign investments to play the diversification/hedging role.

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