ON THE CONVERGENCE RATE OF LEAST－SQUARE ESTIMATORS IN LINEAR REGRESSION MODELS

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摘　要

本文證明在線性迴歸模式中，最小方差估計量的收斂速率。亦即，在適當的條件下，對所有 $\varepsilon>0,\ \alpha>-\frac{1}{2}$，我們有如下的結果：

$$P_r[\sup_{k>n} A_k^{-\alpha} \mid \hat{\beta}_k-\beta \mid >\varepsilon]=0(n^{-p(\alpha+1)+1})$$

ABSTRACT

In this paper we prove the convergence rate result for least－square estimators in linear regression models, i. e. under certain conditions,we have the following result:

$$P_r[\sup_{k>n} A_k^{-\alpha} \mid \hat{\beta}_k-\beta \mid >\varepsilon]=0(n^{-p(\alpha+1)+1})$$

for all $\varepsilon>0,\ \alpha>-\frac{1}{2}$.

1.Introduction

We consider the linear regression model

$$y_i=\alpha+\beta x_i+\varepsilon_i \qquad (i=1, 2,\cdots\cdots) \tag{1}$$

Where $\varepsilon_1, \varepsilon_2, \cdots\cdots$ are i. i. d. random variables with $E\varepsilon_1=0$, $E\varepsilon^2_1=\sigma^2$, $0<\sigma^2<\infty$, and $x_1, x_2, \cdots\cdots$ is an arbitrary sequence of constants, not all equal. The least-square estimate of β based on $x_1, x_2, \cdots\cdots x_n$ is

$$\hat{\beta}_n=\frac{\sum_{i=1}^{n}(x_i-\bar{x}_n)y_i}{\sum_{i=1}^{n}(x_i-\bar{x}_n)^2} \tag{2}$$

Where

$$\bar{x}_n=\frac{1}{n}\sum_{i=1}^{n}x_i$$

From (1), we see that

$$\hat{\beta}_n-\beta=\frac{\sum_{i=1}^{n}(x_i-\bar{x}_n)\varepsilon_i}{\sum_{i=1}^{n}(x_i-\bar{x}_n)^2} \tag{3}$$

so htat

$$E(\hat{\beta}_n-\beta)^2=\sigma^2/A_n \tag{4}$$

Where

$$A_n=\sum_{i=1}^{n}(x_i-\bar{x}_n)^2$$

Hence if the condition

$$\lim_{n\to\infty}A_n=\infty \tag{5}$$

holds, then $\hat{\beta}_n$ converges to β in mean square, and hence in probability, as $n\to\infty$.

Several authors have proved that, under suitable conditions, (5) is a necessary and sufficient condition for $\hat{\beta}_n$ to converge to β with probability one ([1], [2], [3], [4]) .

In section 2, with the magnitude of A_n being restricted to be the ordet of n, we show that $\hat{\beta}_n$ converges to β a. e. Finally, in section 3, we prove the convergence rate of β_n.

2. Almost surely convergence for $\hat{\beta}_n$.

In 1966, Chow [1] has shown the following result:

Lemma 1:

Let ε_n be independent, identical distributed. Then $E\varepsilon_1=0$ and $E\varepsilon^2_1<\infty$, if and only if, for every array a_{nk} of real numbers such that

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_{nk}^2=1, \text{ we have } \lim_{n\to\infty} n^{-1/2} \sum_{k=1}^{n} a_{nk}\, \varepsilon_k=0 \text{ a. e..}$$

By Lemma 1, we obtain the following Theorem 1.

Theorem 1.

$$\text{Suppose} \quad \lim_{n\to\infty} \frac{A_n}{n}=C,\ C>0. \text{ Then}$$

$$\hat{\beta}_n \to \beta, \text{ a. e.}$$

Proof. From (3),

$$\hat{\beta}_n-\beta=A_n^{-1} \sum_{i=1}^{n} (x_i-\bar{x}_n)\, \varepsilon_i$$

$$=A_n^{-1/2} \sum_{i=1}^{n} (x_i-\bar{x}_n)\, \varepsilon_i / A_n^{1/2}.$$

Since

$$\sum_{i=1}^{n} \left[\frac{x_i-\bar{x}_n}{A_n^{1/2}} \right]^2=1 \text{ and } \lim_{n\to\infty} \frac{A_n}{n}=C,$$

by Lemma 1, we see that

$$A_n^{-1/2} \sum_{i=1}^{n} (x_i-\bar{x}_n)\, \varepsilon_i / A_n^{1/2} \to 0 \text{ a. e.,}$$

i. e. $\hat{\beta}_n-\beta\to 0$, a. e.. Therefore, $\hat{\beta}_n \to \beta$, a. e..

3.The rate of convergence

By the Marcinkiewicz－Zygmund Inequality we obtain the following result of the moment of $|\hat{\beta}_n-\beta|$.

Lemma 2.

Let p be a real number $\geqq 2$.

Suppose that $E|\varepsilon_1|^p<\infty$. Then

$$E(|\hat{\beta}_n-\beta|^p)=O(A_n^{-p/2}).$$

Proof. By the Marcinkiewicz－Zygmund Inequality, there exists a B_p (a constant depending only on p) s. t.

$$\begin{aligned}
&E(|\hat{\beta}_n-\beta|^p)\\
&=E(|\sum_{i=1}^{n}\frac{x_i-\bar{x}_n}{A_n}\varepsilon_i|^p)\\
&\leqq B_pE(\sum_{i=1}^{n}\frac{(x_i-\bar{x}_n)^2}{A_n^2}\varepsilon_i^2)^{p/2}\\
&=B_pA_n^{-p}E[\sum_{i=1}^{n}(x_i-\bar{x}_n)^{(2p-4)/p}(x_i-\bar{x}_n)^{4/p}\varepsilon_i^2]^{p/2}\\
&\leqq B_pA_n^{-p}E\{[\sum_{i=1}^{n}(x_i-\bar{x}_n)^2]^{p/2-1}\quad\sum_{i=1}^{n}(x_i-\bar{x}_n)^2|\varepsilon_1|^p\}\\
&=B_pA_n^{-p/2}E|\varepsilon_1|^p\\
&=O(A_n^{-p/2})
\end{aligned}$$

Now, we prove the following convergence rate result for the least－square estimators.

Theorem 2.

Suppose (1) $E|\varepsilon_1|^p<\infty$, for some $p\geqq 2$;

(2) $A_n\approx n^{1+1/(2\alpha+1)}$, for any fixed $\alpha>-\frac{1}{2}$ and $n\geqq 1$.

Then, for all $\varepsilon>0$,

$$P_r[\sup_{k>n} A_k^{-\alpha} \mid \hat{\beta}_k-\beta \mid >\varepsilon]=O(n^{-p(\alpha+1)+1})$$

Proof.

By lemma 2, we have $E \mid \hat{\beta}_n-\beta \mid^p=O(A_n^{-p/2})$, i. e. for some $c>0$, we have

$E \mid \hat{\beta}_k-\beta \mid^p \leqq CA_k^{-p/2},\ k \geqq n.$

$$P_r\{\sup_{k>n} A_k^{-\alpha} \mid \hat{\beta}_k-\beta \mid >\varepsilon\}$$

$$\leqq \sum_{k=n+1}^{\infty} P_r\{A_k^{-\alpha} \mid \hat{\beta}_k-\beta \mid >\varepsilon\}$$

$$\leqq \sum_{k=n+1}^{\infty} \varepsilon^{-p} A_k^{-p\alpha} E \mid \hat{\beta}_k-\beta \mid^p$$

$$\leqq \sum_{k=n+1}^{\infty} \varepsilon^{-p} A_k^{-p\alpha} CA_k^{-p/2}$$

$$=C\varepsilon^{-p} \sum_{k=n+1}^{\infty} A_k^{-p(2\alpha+1)/2}$$

$$\approx C\varepsilon^{-p} \sum_{k=n+1}^{\infty} (k^{1+1/(2\alpha+1)})^{-p(2\alpha+1)/2}$$

$$=C\varepsilon^{-p} \sum_{k=n+1}^{\infty} k^{-p(\alpha+1)}$$

$$=O(n^{-p(\alpha+1)+1})$$

Corollary 1.

Under the assumptions of Theorem 2. Then for all $\varepsilon>0$ and $\delta>0$ we have

$$\sum_{n=1}^{\infty} n^{p\alpha-\delta} P_r[\sup_{k>n} A_k^{-\alpha} \mid \hat{\beta}_k-\beta \mid >\varepsilon]<\infty.$$

Corollary 2.

Under the conditions of corollary 1, we have

$$P_r[\sup_{k>n} A_k^{-\alpha} \mid \hat{\beta}_k-\beta \mid >\varepsilon]=o(n^{-(p\alpha+1)+\delta})$$

Reference

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