# THE POSITION AND VELOCITY ESTIMATION ERRORS OF MONEY SUPPLY IN WIENER FILTERING STATE SPACE WITH UNCERTAIN PARAMETERS

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## 摘　要

本文利用 Kalman-Schmidt (1963, 1966) 之 Wiener 濾淨狀態空間不確定參數方法，來模擬估計貨幣供給過程中之位置與變動速度之動態誤差。由此研究，可更清楚了解到含有衡量誤差之貨幣供給之動態估計變化性質，亦可獲得一些有關 CGRR 政策之有趣重要命題。

## Abstract

We use Kalman-Schmidt's (1963, 1966) idea to measure the position and velocity estimation errors of money supply when measurement-errors uncertainty exists in the money supply process. Several interesting propositions about the constant growth rate rule (CGRR) of money supply policy are proposed in this paper.

## 1. Introduction

This note is intended to present some detailed computations and analytical comparisons of dynamic estimation errors for the position versus velocity of two types of change in money supply suffering from measurement noise, i.e., constant money supply and incereasing growth rate of money supply, in Wiener filtering state space with uncertain parameters. Our idea originates from the analytical examples in Kalman (1963) and a numerical example in Schmidt (1966). The

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detailed solution of our proposed problem gives us an illustrative "feel" for how the dynamic estimation errors of money supply will evolve in an economy with uncertain parameters and an understanding for how the Wiener filtering method can be meaning fully applied to an economic example.

## 2. The Model

Our problem consists of a second-order system representing two types of change in money supply in a constant field.

(1) $\ddot{M}_t = g, \; t \geqslant 0$

Where $M_t \equiv$ the position of money supply at t, $M_t \equiv$ the accelerating change of money supply at t, $g \equiv 0$ (if the authority adopts the constant money supply policy), $g > 0$ (a positive constant if the authority adopts an increasing growth rate of money supply policy).

Let the position of $M_t$ be $M_t = (x_t)_1$, and the velocity of $M_t$ be $\dot{M}_t = (x_t)_2$.

Then, defining the vector $X_t$ as

(2) $X_t^T \equiv [\,(x_t)_1, (x_t)_2\,]$,

(1) can be written in system form

(3) $\dot{X}_t = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} X_t + \begin{bmatrix} 0 \\ g \end{bmatrix}$, where $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \triangleq F$

It is easy to verify that the state transition matrix of this system is

(4) $\Phi(t, \tau) = \begin{bmatrix} 1 & t-\tau \\ 0 & 1 \end{bmatrix}$

so that

(5) $X_t = \Phi(t, \tau) X_\tau + \int_\tau^t \begin{bmatrix} 1 & t-s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ g \end{bmatrix} ds$

$= \Phi(t, \tau) X_\tau + g \begin{bmatrix} (t-\tau)^2/2 \\ t-\tau \end{bmatrix}$

Therefore,

$$(6)\quad X_{t+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} X_t + g \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \quad t = 0, 1, 2, \ldots\ldots\ldots$$

or alternatively,

$$(7a)\quad M_{t+1} = M_t + \dot{M}_t + \frac{1}{2} g,$$

$$(7b)\quad \dot{M}_{t+1} = \dot{M}_t + g = M_{t+1} - M_t + \frac{1}{2} g$$

## 3. Observability and Simulation of the Model

We will take scalar observations of position of money supply

$$(8)\quad Y_t = [\,1 \quad 0\,]\, X_t + V_t\ ;\ t = 1, \ldots\ldots, 6, \text{ Where } [\,1 \quad 0\,] \triangleq U$$

with measurement noise $V_t \sim N(0, R(t))$ where the variance of the noise is assumed to be constant and be equal to one for the convenience of doing simulation, i.e., $R(t) = 1$. We assume that $X_0 \sim N(\hat{X}_0, P_0)$, where

$$(9)\quad \hat{X}_0 = \begin{bmatrix} 95 \\ 1 \end{bmatrix}, \quad P_0 = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}$$

Let us first show that our model is uniformly completely observable.

Proof:

Because the value of information matrix here

$$(10)\quad \left| \mathscr{I}(t, t\text{-}1) \right| = \left| \sum_{i=t-1}^{t} \Phi^T(i, t) U^T(i) R^{-1}(i) U(i) \Phi(i, t) \right|$$

is computed to be

$$(11)\quad 0 < \left| \alpha I \right| \leqslant \left| \mathscr{I}(t, t\text{-}1) \right| = \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} \leqslant \left| \beta I \right| \text{ (Take } \alpha = 0.1, \beta = 3, \text{ for example)}$$

Therefore, the discrete dynamical system (6), (8) is uniformly completely observable.

Next we do the following simulation for the model. Take the positive constant $g = 0.1$ (10%) in (1). Assume that the "true" initial condition of money supply is given by a constant specified level, e.g., $M_{-1} = M_0 = 100$

and $M_0 = 0$, for the convenience of simulation.

(12) $(X_0)_1 = M_0 = 100,\ (X_0)_2 = \dot{M}_0 = 0$

Then the true trajectory (at t = 1, . . . . , 6) and a realization of the observations for one type of the increasing growth rate of money supply are given in Table 1. We show the details of one cycle of the filter computations as follows.

**Step 1 Compute the predicted state by**

(13) $$\hat{X}(1|0) = \Phi(1,0)\ \hat{X}(0|0) + g \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 95 \\ 1 \end{bmatrix} + \begin{bmatrix} 0.05 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 96.05 \\ 1.1 \end{bmatrix}$$

**Step 2 Compute the predicted error covariance matrix by**

(14) $$P(1|0) = \Phi(1,0)\ P(0|0)\ \Phi^T(1,0)$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 1 \\ 1 & 1 \end{bmatrix}$$

**Step 3 Compute the filter gain matrix by**

(15) $$K(1) = P(1|0)\ U^T(1)\ [\ U(1) P(1|0)\ U^T(1) + R(1)\ ]^{-1}$$

$$= \begin{bmatrix} 11 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left[ [\ 1\ \ 0\ ] \begin{bmatrix} 11 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \right]^{-1}$$

$$= \frac{1}{12} \begin{bmatrix} 11 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.92 \\ 0.08 \end{bmatrix}$$

**Step 4 Process the observation Yt+1 by**

(16) $$\hat{X}(1|1) = \hat{X}(1|0) + K(1)\ [\ Y(1) - U(1)\ \hat{X}(1|0)\ ]$$

$$= \begin{bmatrix} 96.05 \\ 1.1 \end{bmatrix} + \begin{bmatrix} 0.92 \\ 0.08 \end{bmatrix} [\ 100 - 96.05\ ] = \begin{bmatrix} 99.684 \\ 1.416 \end{bmatrix}$$

**Step 5 Compute the new error covariance matrix by**

(17) $P(1|1) = [ I - K(1)\,U(1) ]\, P(1|0)$

$$= \left[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.92 \\ 0.08 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right] \begin{bmatrix} 11 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.08 & 0 \\ -0.08 & 1 \end{bmatrix} \begin{bmatrix} 11 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.88 & 0.08 \\ 0.12 & 0.92 \end{bmatrix}$$

**Step 6 Forward time dating one period, and return to step 1.**

Alternatively, take $g \equiv 0$ (0%) in (1). We also assume the "true" initial condition of money supply is the same as (12). Then the true trajectory (at $t = 1 \ldots\ldots\ldots 6$) and a realization of the observations for the type of the constant growth rate of money supply are given in Table 2. We also show one cycle of the filter computations as follows.

**Step 1 Compute the predicted state by**

(18) $\hat{X}(1|0) = \Phi(1, 0)\, \hat{X}(0|0)$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 95 \\ 1 \end{bmatrix} = \begin{bmatrix} 96 \\ 1 \end{bmatrix}$$

**Step 2 Compute the predicted error covariance matrix by**

(19) $P(1|0) = \Phi(1, 0)\, P(0|0)\, \Phi^T(1, 0)$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 1 \\ 1 & 1 \end{bmatrix}$$

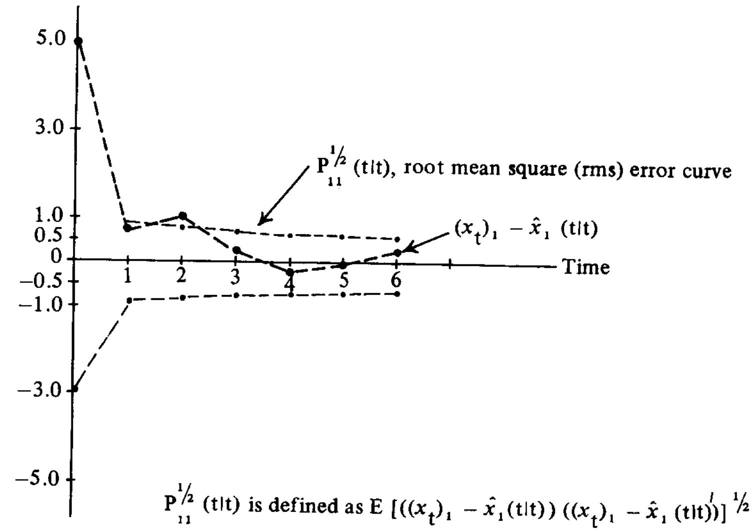
Table 1 Dynamic Estimation Errors of Money Supply with an Increasing Growth Rate of 10% (g = 0.1) When Measurement Errors Exist

| t | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $(x_t)_1 = M_t$ | 100.0 | 100.5 | 101.55 | 101.8 | 102.15 | 102.6 | 103.15 |
| $(x_t)_2 = M_t$ | 0. | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 |
| $V_t$ | | –0.5 | –1.5 | –0.25 | –0.35 | –0.45 | –0.55 |
| $Y_t$ | | 100.0 | 100.5 | 101.55 | 101.8 | 102.15 | 102.6 |
| $\hat{x}_1\ (t\|t) = \hat{M}_t\ (t\|t)$ | 95.0 | 99.684 | 100.7145 | 101.72359 | 102.26863 | 102.696633543 | 103.149742138 |
| $\hat{x}_2\ (t\|t) = \hat{M}_t\ (t\|t)$ | 1.0 | 1.416 | 1.29045 | 1.23093 | 1.0480448 | 0.93917317572 | 0.89970599961 |
| $P_{11}\ (t\|t)$ | 10.0 | 0.88 | 0.66 | 0.65624 | 0.6109404 | 0.5507156083 | 0.49370254092 |
| $P_{22}\ (t\|t)$ | 1.0 | 0.92 | 0.573 | 0.28773 | 0.1472517 | 0.0820145913 | 0.04955890683 |
| $(x_t)_1 - \hat{x}_1\ (t\|t)$ | 5.0 | 0.816 | 0.8355 | 0.07641 | –0.11863 | –0.09663 | 0.00026 |
| $(x_t)_2 - \hat{x}_2\ (t\|t)$ | –1.0 | –1.316 | –1.09045 | –0.93093 | –0.6480448 | –0.4391731 | –0.2997059 |
| $P_{11}^{1/2}\ (t\|t) = \sigma_{11}$ | 3.1622776 | 0.9380831 | 0.8124038 | 0.8100864 | 0.7816267 | 0.7421018 | 0.7026396 |
| $P_{22}^{1/2}\ (t\|t) = \sigma_{22}$ | 1.0 | 0.9591633 | 0.7569676 | 0.5364046 | 0.3837338 | 0.2863937 | 0.2226182 |

Table 2 Dynamic Estimation of Money Supply with Constant Growth Rate of 0% (g=0) When Measurement Errors Exist

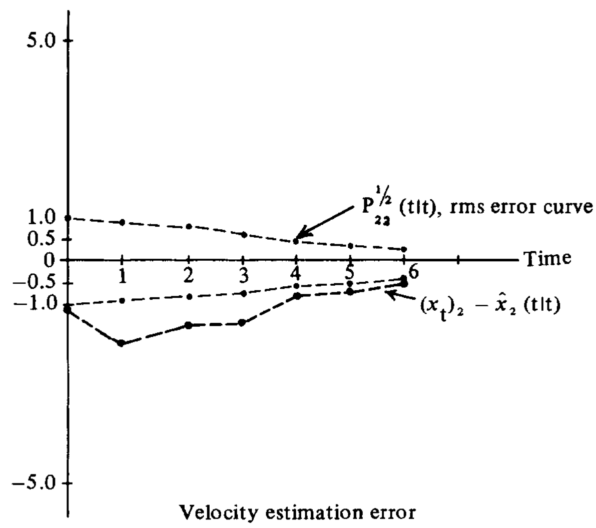
| t | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $(x_t)_1 = M_t$ | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 |
| $(x_t)_2 = M_t$ | 0. | 0. | 0. | 0. | 0. | 0. | 0. |
| $V_t$ | | 0. | 0. | 0. | 0. | 0. | 0. |
| $Y_t$ | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 |
| $\hat{x}_1\ (t\|t) = \hat{M}_t\ (t\|t)$ | 95.0 | 99.68 | 100.33 | 100.447920583 | 100.39268 | 100.322137231 | 100.264983564 |
| $\hat{x}_2\ (t\|t) = \hat{M}_t\ (t\|t)$ | 1.0 | 1.32 | 0.973 | 0.561361452 | 0.3243284 | 0.20123802728 | 0.13401285616 |
| $P_{11}\ (t\|t)$ | 10.0 | 0.88 | 0.66 | 0.6562397 | 0.6109404 | 0.55071516083 | 0.49370254092 |
| $P_{22}\ (t\|t)$ | 1.0 | 0.92 | 0.573 | 0.2877279 | 0.1472517 | 0.08202145913 | 0.04955890683 |
| $(x_t)_1 - \hat{x}_1\ (t\|t)$ | 5.0 | 0.32 | –0.33 | –0.447920583 | –0.39269 | –0.322137231 | –0.264983564 |
| $(x_t)_2 - \hat{x}_2\ (t\|t)$ | –1.0 | –1.32 | –0.973 | –0.561361452 | –0.3243284 | –0.20123802728 | –0.13401285616 |
| $P_{11}^{1/2}\ (t\|t) = \sigma_{11}$ | 3.1622776 | 0.9380831 | 0.8124038 | 0.8100864 | 0.7816267 | 0.7421018 | 0.7026396 |
| $P_{22}^{1/2}\ (t\|t) = \sigma_{22}$ | 1.0 | 0.9591663 | 0.7569676 | 0.5364046 | 0.3837338 | 0.2863937 | 0.2226182 |

The Position and Velocity Estimation Errors of Money Supply in Wiener Filtering State Space with Uncertain Parameters

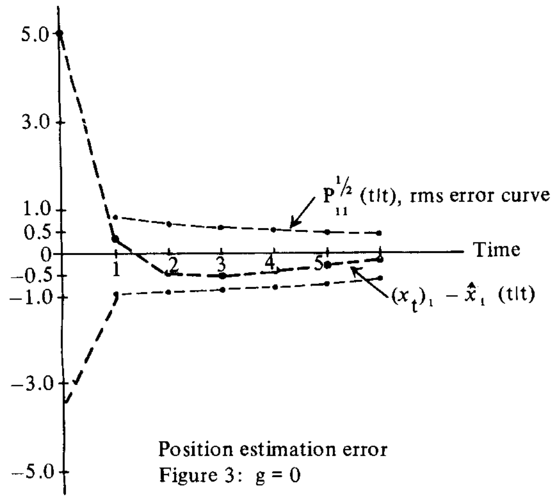



$P_{11}^{1/2}$ (t|t) is defined as E $[((x_t)_1 - \hat{x}_1(t|t))((x_t)_1 - \hat{x}_1(t|t))']^{1/2}$

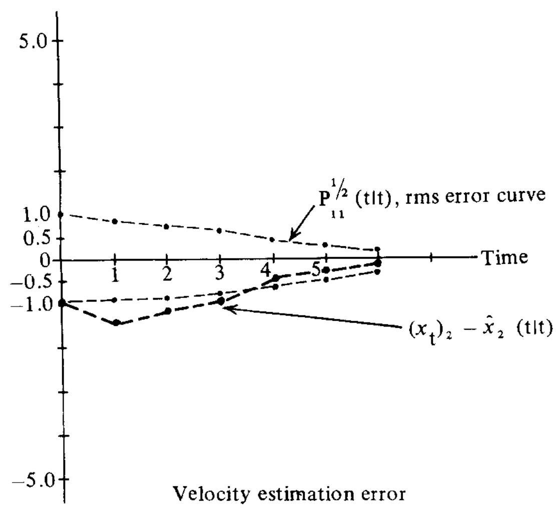
Position estimation error
Figure 1: g = 0.1 (10%)




Velocity estimation error
Figure 2: g = 0.1 (10%)




Position estimation error
Figure 3: g = 0




Velocity estimation error
Figure 4: g = 0

**Step 3 Compute the filter gain matrix by**

(20) $K(1) = P(1|0)\, U^T(1)\, [\, U(1)\, P(1|0)\, U^T(1) + R(1)\, ]^{-1}$

$$= \begin{bmatrix} 11 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left[ [\,1 \quad 0\,] \begin{bmatrix} 11 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \right]^{-1}$$

$$= \frac{1}{12} \begin{bmatrix} 11 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.92 \\ 0.08 \end{bmatrix}$$

**Step 4 Process the observation $Y_{t+1}$ by**

(21) $\hat{X}(1|1) = \hat{X}(1|0) + K(1)\, [\, Y(1) - U(1)\, \hat{X}(1|0)\, ]$

$$= \begin{bmatrix} 96 \\ 1 \end{bmatrix} + \begin{bmatrix} 0.92 \\ 0.08 \end{bmatrix} [\, 100 - 96\, ] = \begin{bmatrix} 99.68 \\ 1.32 \end{bmatrix}$$

**Step 5 Compute the new error covariance matrix by**

(22) $P(1|1) = [\, I - K(1)\, U(1)\, ]\, P(1|0)$

$$= \left[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.92 \\ 0.08 \end{bmatrix} [\,1 \quad 0\,] \right] \begin{bmatrix} 11 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.08 & 0 \\ -0.08 & 1 \end{bmatrix} \begin{bmatrix} 11 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.88 & 0.08 \\ 0.12 & 0.92 \end{bmatrix}$$

**Step 6 Forward time dating one period, and return to step 1.**

## 4. Graphical Depiction of the Filtering Results

The filtering results we derived in section 3 are shown in Figs. 1-4. Plotted in these figures are the dynamic estimation errors of the position and velocity for money supply with an increasing growth rate of 10% ($\dot{M}_t$ = gt) and with a constant growth rate of 0% ($M_t$ = 0) respectively. The dashed lines with light ink represent the root mean square (rms) errors. The dashed lines with heavy ink represent the estimation errors.

From our particular simulation, we find the following propositions.

**Proposition 1:**

The velocity estimation errors of money supply are less likely to fall inside the rms curves than the position estimation errors of money supply for both cases of g = 0% and g = 10%.

**Proposition 2:**

Given the same initial conditions of money supply, both the velocity estimation errors and the position estimation errors at g = 0% are more likely to fall inside the rms errors curves than those errors at g = 10%.

**Proposition 3:**

For both cases of g = 0% and g = 10%, the rms position error drops dramatically as soon as the first observation is processed, but the rms velocity error does not decrease substantially until the second observation is processed. The reason for this proposition is because two position observations are required to determine both components of the state vector. The dynamics of our money supply system are such that velocity affects the position, whereas position does not affect velocity. As a consequence, velocity must first be estimated rather accurately before good estimates of position can be had. This explains the fact that after the second observation the velocity rms errors steadily decrease, whereas position rms errors decrease much more slowly. However, both rms errors will, of course, go to zero eventually.

## 5. Implications

From the above illustration, we have learned that even the monetary authority adopts a constant growth rate rule (CGRR) of money supply policy, we still suffer from some dynamic prediction errors of position and velocity if the measurement-errors uncertainty exists in the money supply process. Moreover, we have also found that the larger constant growth rate of money supply is, the larger dynamic prediction errors will be, and thus the less probability of its falling inside the rms errors curves will happen.

These findings enhance our belief that the monetary authority sticks to a CGRR policy deserves encouragement, otherwise the dynamic predicion errors of position and velocity would become larger if the policy introduces other uncertainties into the economy except the measurement errors of money supply.

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