A Pedagogical Chapter on the Dynamic Stability of IS-LM Model

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摘　　要

本文對總體 IS-LM 模型之動態穩定分析，提供一個啓發性的動態調整分析，並澄清一些總體經濟學對 IS-LM 動態穩定分析之誤解。尤其是當 IS 斜率爲正，且較 LM 平坦時，大部份的總體經濟學著作皆犯有錯誤。因此，本文應有其學術價值與可參考之處。

ABSTRACT

This paper intends to render a heuristic supplement to the dynamic stability analysis of IS-LM model. We clarify and enter every possible dynamic adjustment path into details which was not examined in the literature before.

It is usually wrong and misleading for some popular modern macroeconomics to regard the conventional IS-LM model as dynamically stable in case that IS is positively sloped and when it is flatter than LM. Our paper renders a worthwhile clarification of this long-existing misunderstanding of the IS-LM dynamic stability analysis.

I. Introduction

Most popular macroeconomics textbooks such as Ackley (1978, pp. 380-382), Bear (1978, p. 365), Branson (1979, pp. 236-239), Dernburg and Dernburg (1969, pp. 227-232), Dernburg and Mcdougall (1972, pp. 302-308), Hadjimichalakis (1982, pp. 232-235), Kelly (1981, pp. 87-89), Kuo (1978, pp. 191-194, in Chinese), Lin (1978, p. 30, in Chinese), Meyer (1980, pp. 276-284), Ott-Ott-Yoo (1975, pp. 39-42), Shih (1984, pp. 243-262, in Chinese), Smith (1970, pp. 278-279), . . . , etc.

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make the dynamic adjustment path of IS-LM model very ambiguous and some even confuse the stability property, which may be due to ignorance or the limitation of space.

This paper intends to render a heuristic supplement to the dynamic stability analysis of IS-LM model. Although we innovate nothing new about the frontier of the dynamic stability theorem, however, we clarify and enter every possible dynamic adjustment path into datails which was not examined in the literature before.

II. The Characteristic Equation of the Illustrated Model

A conventional macromodel considered here is

I(i, y)–S(y) = 0 . . . . IS equililrium locus (1)

L(i, y) – M^s = 0 . . . . LM equililrium locus (2)

where I = investment, S = saving, L = demand for money, M^s = fixed money supply, i = interest rate, y = real income.

The dynamic adjustment equations for this model are specified as usual:

$$\frac{dy}{dt} = k_1 [I(i, y) - S(y)],\ 0 \leqslant k_1 \leqslant \infty \tag{3}$$

$$\frac{di}{dt} = k_2 [L(i, y) - M^s],\ 0 \leqslant k_2 \leqslant \infty \tag{4}$$

Taking Taylor's linear approximation for equations (3) and (4) around the initial equilibrium point (y*, i*) yields

$$\frac{dy}{dt} = k_1 (I_y - S_y)(y - y^*) + k_1 I_i (i - i^*) \tag{5}$$

$$\frac{di}{dt} = k_2 L_y (y - y^*) + k_2 L_i (i - i^*) \tag{6}$$

where $I_y \equiv \frac{\partial I}{\partial y} > 0,\ S_y \equiv \frac{\partial S}{\partial y} > 0,\ L_y \equiv \frac{\partial L}{\partial y} > 0,\ I_i \equiv \frac{\partial I}{\partial i} < 0,\ L_i \equiv \frac{\partial L}{\partial i} < 0.$

Letting the general solution be $y = y^* + \alpha e^{\lambda t}$, and $i = i^* + \beta e^{\lambda t}$ where α and β are nonzero constants, the above system then becomes:

$$\begin{bmatrix} k_1 (I_y - S_y) - \lambda & k_1 I_i \\ k_2 L_y & k_2 L_i - \lambda \end{bmatrix} \begin{bmatrix} \alpha e^{\lambda t} \\ \beta e^{\lambda t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7}$$

To find values for λ such that equation (7) yields nonzero α and β, we derive the following auxiliary equation:

$$\lambda^2 - [k_1(I_y - S_y) + k_2 L_i]\lambda + k_1 k_2 [L_i(I_y - S_y) - I_i L_y] = 0 \tag{8}$$

with the roots λ_1 and λ_2 satisfying:

$$\lambda_1 + \lambda_2 = k_1(I_y - S_y) + k_2 L_i \tag{9-1}$$

$$\lambda_1 \lambda_2 = k_1 k_2 [L_i(I_y - S_y) - I_i L_y] \tag{9-2}$$

For the simplicity of the following presentation, we set $\Delta = [k_1(I_y - S_y) + k_2 L_i]^2 - 4k_1 k_2 [L_i(I_y - S_y) - I_i L_y]$.

III. Dynamic Perspective on the Types of Equilibrium Point

Case 1. λ_1 and λ_2 are real, distinct, and of the same sign.

For convenience, we may assume that $\lambda_1 > \lambda_2$. Each solution pair has the form

$$y(t) = y^* + c_1 \alpha_1 e^{\lambda_1 t} + c_2 \alpha_2 e^{\lambda_2 t} \tag{10-1}$$

$$i(t) = i^* + c_1 \beta_1 e^{\lambda_1 t} + c_2 \beta_2 e^{\lambda_2 t} \tag{10-2}$$

where c_1 and c_2 are arbitrary constants. Further analysis as regards positive and negative roots is shown below:

Case 1 (a) Both roots are negative, i.e., $\lambda_2 < \lambda_1 < 0$.

Clearly this condition makes all solutions tend to (y^*, i^*) as t tends to ∞. That is

$$\lim_{t \to \infty} y(t) = y^* \quad and \quad \lim_{t \to \infty} i(t) = i^* \tag{11}$$

There exists the following three possibilities to occur:

(1) $c_1 = 0$ and $c_2 \neq 0$, then

$$i(t) - i^* = \frac{\beta_2}{\alpha_2} [y(t) - y^*] \tag{12}$$

This implies that the orbit is a straight line with the slope $\frac{\beta_2}{\alpha_2}$.

(2) $c_1 \neq$ and $c_2 = 0$, then

$$i(t) - i^* = \frac{\beta_1}{\alpha_1} [y(t) - y^*] \tag{13}$$

This shows that the orbit is a straight line with the slope $\frac{\beta_1}{\alpha_1}$.

(3) $c_1 \neq 0$ and $c_2 \neq 0$, then

$$\frac{i(t)-i^*}{y(t)-y^*}=\frac{c_1\beta_1+c_2\beta_2 e^{(\lambda_2-\lambda_1)t}}{c_1\alpha_1+c_2\alpha_2 e^{(\lambda_2-\lambda_1)t}} \tag{14-1}$$

and

$$\lim_{t\to\infty}\frac{i(t)-i^*}{y(t)-y^*}=\frac{\beta_1}{\alpha_1} \tag{14-2}$$

$$\lim_{t\to-\infty}\frac{i(t)-i^*}{y(t)-y^*}=\frac{\beta_2}{\alpha_2} \tag{14-3}$$

Thus all orbits approach the equilibrium point (y*, i*) with the slope $\frac{\beta_1}{\alpha_1}$ as t tends to ∞, and with the other slope $\frac{\beta_2}{\alpha_2}$ as t tends to - ∞. Apparently, the equilibrium solution (y*, i*) is asymptotically stable, and it is called *a stable node.*

The conditions for case 1 (a) to happen require that the coefficients in characteristic equation (8) satisfy: 1. The slope of LM be larger than that of IS, 2. IS may slope either upward ($I_y > S_y$) or downward ($I_y < S_y$), 3. The relative speed of dymanic adjustment, $\frac{k_2}{k_2}$, should be larger than $\frac{I_y - S_y}{-L_i}$ for $I_y > S_y$, and 4. $\Delta > 0$. The graph for $I_y < S_y$ is depicted in Fig. 1-1, 1-2, and 1-3 for three different values of the slopes $\frac{\beta_1}{\alpha_1}$ and $\frac{\beta_2}{\alpha_2}$. The graph for $I_y > S_y$ is left as an exercise for interested reader.

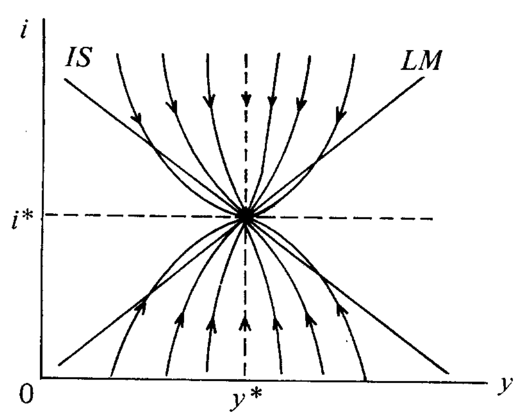

Fig. 1-1

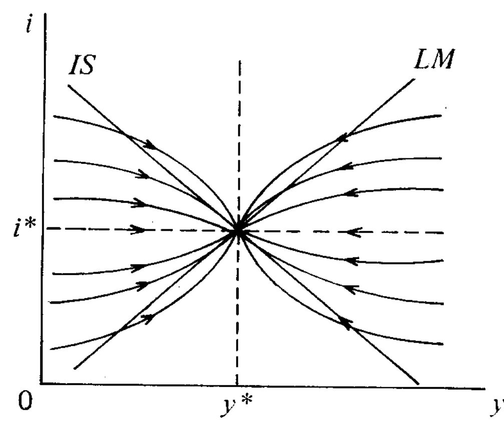

Fig. 1-2

Fig. 1-1. $\frac{\beta_1}{\alpha_1}=0, \frac{\beta_2}{\alpha_2}=\infty, S_y > I_y$

Fig. 1-2. $\frac{\beta_1}{\alpha_1}=\infty, \frac{\beta_2}{\alpha_2}=0, S_y > I_y$

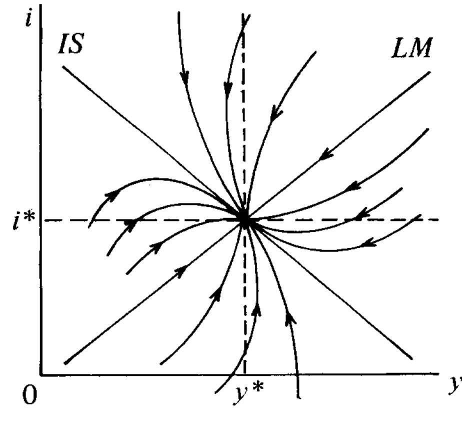

Fig. 1-3

Fig. 1-3. $\frac{\beta_1}{\alpha_1}$ = *IS* slope, $\frac{\beta_2}{\alpha_2}$ = *LM* slope, $S_y > I_y$

Case 1 (a) Stable node ($\lambda_1 = \lambda_2$, $\lambda_2 < \lambda_1 < 0$)

Case 1 (b) Both roots are positive, i.e., $\lambda_1 > \lambda_2 > 0$.

Now all solutions except the (y*, i*) point approach ∞ as t tends to ∞ because we obtain

$$\lim_{t \to \infty} y(t) = y^* + \infty \tag{15-1}$$

$$\lim_{t \to \infty} i(t) = i^* + \infty \tag{15-2}$$

Hence the equilibrium solution is unstable, and it is called *an unstable node.* The orbits are the same as those in case 1 (a), except that the direction of motion is reversed. As t tends to $-\infty$, the orbits approach (y*, i*) with the slope $\frac{\beta_2}{\alpha_2}$, and as t tends to ∞, the orbits become asymptotic to the line with the slope $\frac{\beta_1}{\alpha_1}$.

The conditions for case 1 (b) to appear require the coefficients in equation (8) satisfy: 1. LM be steeper than IS, 2. IS should slope upward ($I_y > S_y$), 3. $\frac{k_2}{k_1} < \frac{I_y - S_y}{-L_i}$, and 4. $\Delta > 0$. The graphical depiction is illustrated in Fig. 1-4, 1-5, and 1-6 for three different values of the slopes $\frac{\beta_1}{\alpha_1}$ and $\frac{\beta_2}{\alpha_2}$.

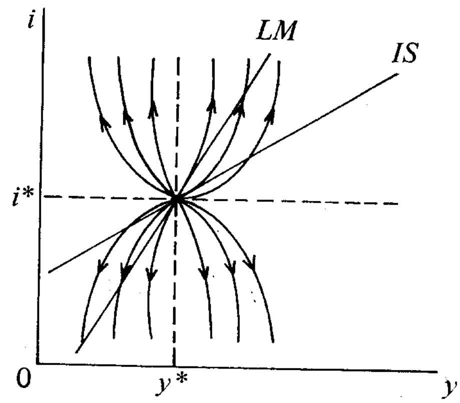

Fig. 1-4

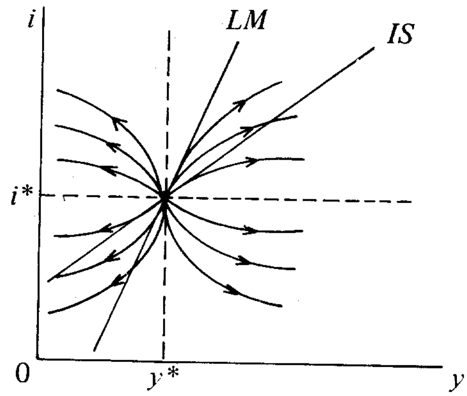

Fig. 1-5

Fig. 1-4. $\frac{\beta_1}{\alpha_1} = 0, \frac{\beta_2}{\alpha_2} = \infty, S_y < I_y$

Fig. 1-5. $\frac{\beta_1}{\alpha_1} = \infty, \frac{\beta_2}{\alpha_2} = 0, S_y < I_y$

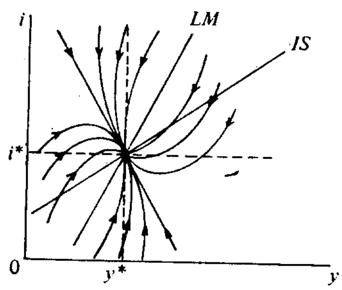

Fig. 1-6

Fig. 1-6. $\frac{\beta_1}{\alpha_1} = -\frac{4}{3}, \frac{\beta_2}{\alpha_2} = LM$ slope, $S_y < I_y$

Case 1 (b) unstable node ($\lambda_1 \neq \lambda_2, \lambda_1 > \lambda_2 > 0$)

Case 2 Both roots are real with opposite signs, i.e., $\lambda_1 > 0 > \lambda_2$.

We analyze the following three possibilities:

(1) $c_1 = 0$ *and* $c_2 \neq 0$, *then*

$$\lim_{t\to\infty} y(t) = y^*, \quad \lim_{t\to-\infty} y(t) = y^* + \infty \tag{16-1}$$

$$\lim_{t\to\infty} i(t) = i^*, \quad \lim_{t\to-\infty} i(t) = i^* + \infty \tag{16-2}$$

$$i(t) - i^* = \frac{\beta_2}{\alpha_2} [y(t) - y^*] \tag{16-3}$$

As t tends to ∞, (y(t), i(t)) will approach the equilibrium point (y*, i*), however, as t tends to $-\infty$, it approaches infinity.

(2) $c_1 \neq 0$ and $c_2 = 0$, then

$$\lim_{t\to\infty} y(t) = y^* + \infty, \quad \lim_{t\to-\infty} y(t) = y^* \tag{17-1}$$

$$\lim_{t\to\infty} i(t) = i^* + \infty, \quad \lim_{t\to-\infty} i(t) = i^* \tag{17-2}$$

$$i(t) - i^* = \frac{\beta_1}{\alpha_1}[y(t) - y^*] \tag{17-3}$$

Both y(t) and i(t) approach ∞ as t tends to ∞, and approach (y^*, i^*) as t tends to $-\infty$. Two orbits for (16.3) and (17.3) are sketched in Fig. 2-1.

(3) $c_1 \neq 0$ and $c_2 \neq 0$, then the system is more complicated and we have:

$$\frac{i(t)-i^*}{y(t)-y^*} = \frac{c_1\beta_1 + c_2\beta_2 e^{(\lambda_2-\lambda_1)t}}{c_1\alpha_1 + c_2\alpha_2 e^{(\lambda_2-\lambda_1)t}}$$

$$= \frac{c_1\beta_1 e^{(\lambda_1-\lambda_2)t} + c_2\beta_2}{c_1\alpha_1 e^{(\lambda_1-\lambda_2)t} + c_2\alpha_2} \tag{18-1}$$

$$\lim_{t\to\infty} \frac{i(t)-i^*}{y(t)-y^*} = \frac{\beta_1}{\alpha_1}, \quad \lim_{t\to-\infty} \frac{i(t)-i^*}{y(t)-y^*} = \frac{\beta_2}{\alpha_2} \tag{18-2}$$

$$\lim_{t\to\pm\infty} y(t) = y^* + \infty, \quad \lim_{t\to\pm\infty} i(t) = i^* + \infty \tag{18-3}$$

These results imply that all orbits are asymptotic to the line with the slope $\frac{\beta_1}{\alpha_1}$ as t tends to ∞, and to another line with the slope $\frac{\beta_2}{\alpha_2}$ as tends to $-\infty$. Both y(t) and i(t) approach ∞ as t tends to $\pm\infty$. It is clear that (y^*, i^*) point is unstable, and it is called *a saddle point* with the property that exactly one orbit approaches the solution and all others are repelled by it.

The conditions for this case to happen require that the coefficients in equation (8) satisfy

$$k_1(I_y - S_y) + k_2 L_i \lesseqgtr 0 \text{ as } \lambda_1 \lesseqgtr |\lambda_2| \tag{19-1}$$

$$k_1 k_2 [L_i(I_y - S_y) - I_i L_y] < 0 \tag{19-2}$$

$$\Delta > 0 \tag{19-3}$$

which implies that 1. $I_y > S_y$, 2. IS slopes upward, 3. IS is steeper than LM, and 4.

$\frac{k_2}{k_1} \lessgtr \frac{I_y - S_y}{-L_i}$ as $\lambda_1 \gtrless |\lambda_2|$. The phase diagram corresponding to such a saddle point is depicted in Fig. 2-2.

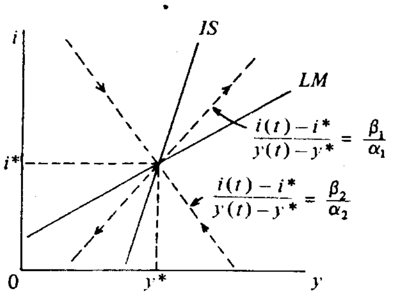

Fig. 2-1

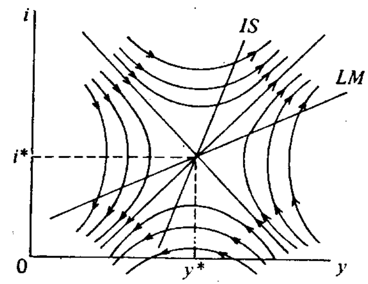

Fig. 2-2

Fig. 2-1. $I_y > S_y$, IS slope $>$ LM slope Fig. 2-2. $I_y > S_y$, IS slope $>$ LM slope

Case 2 Saddle point ($\lambda_1 > 0 > \lambda_2$)

Case 3 Both roots λ_1 and λ_2 are real, equal, and of the same sign.

We now discuss its dynamic adjustment path according to both situations of $\lambda < 0$ and $\lambda > 0$ as follows:

Case 3 (a) $\lambda_1 = \lambda_2 = \lambda < 0$. There esists two possib i lities by which the auxiliary equation (8) may yield a double root. One is $k_1(I_y - S_y) = k_2 L_i \neq 0$, and $k_1 I_i = k_2 L_y = 0$ such that equation (8) becomes

$$\lambda^2 - 2k_1(I_y - S_y)\lambda + k_1^2(I_y - S_y)^2 = 0 \tag{20-1}$$

or

$$\lambda^2 - 2k_2 L_i \lambda + k_2^2 L_i^2 = 0 \tag{20-2}$$

Then we have the double root, $\lambda = k_1(I_y - S_y) = k_2 L_i$. The solutions for y(t) and i(t) are obviously of the form:

$$y(t) = y^* + c_1 e^{\lambda t}, \quad \lim_{t \to \infty} y(t) = y^* \tag{21-1}$$

$$i(t) = i^* + c_2 e^{\lambda t}, \quad \lim_{t \to \infty} i(t) = i^* \tag{21-2}$$

and

$$\frac{i(t) - i^*}{y(t) - y^*} = \frac{c_2}{c_1} \tag{21-3}$$

All orbits are straight lines with the slope $\frac{c_2}{c_1}$, and all solutions will asymptotically approach (y*, i*) stably as t tends to ∞ since $\lambda<0$. It is called *a proper node* or *a star-shaped node.* The conditions for this situation to exist require that $I_y < S_y$, $\Delta=0$, and IS be vertical and LM be horizontal due to $I_i = L_y = 0$. This diagram is graphed in Fig. 3-1.

Another one is $k_1(I_y - S_y) \neq k_2 L_i \neq 0$, and $k_2 L_y \neq k_1 I_i \neq 0$, then the equations are coupled and the general solution is

$$y(t) = y^* + [c_1\alpha_1 + c_2(\alpha_2 + \alpha_3 t)]e^{\lambda t} \tag{22-1}$$

$$i(t) = i^* + [c_1\beta_1 + c_2(\beta_2 + \beta_3 t)]e^{\lambda t} \tag{22-2}$$

with

$$\lim_{t\to\infty} y(t) = y^* \tag{22-3}$$

$$\lim_{t\to\infty} i(t) = i^* \tag{22-4}$$

$$\lim_{t\pm\infty} \frac{i(t)-i^*}{y(t)-y^*} = \frac{\beta_3}{\alpha_3} \tag{22-5}$$

All orbits are asymptotic to the line with the slope $\frac{\beta_3}{\alpha_3}$ as t tends to ±∞, and actual (y(t), i(t)) will asymptotically approach the equilibrium point (y*, i*) stably as t tends to ∞. It is called *a stable node.* The conditions for this existence require that $\Delta=0$, LM be steeper than IS, and IS may slope either upward ($I_y > S_y$) or downward ($I_y < S_y$). The adjustment path for the case of $I_y > S_y$ is graphed in Fig. 3-2.

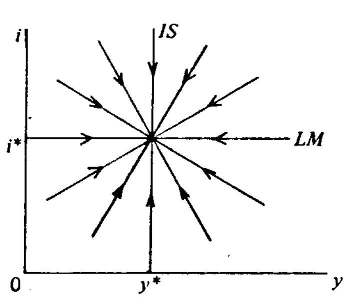

Fig. 3-1

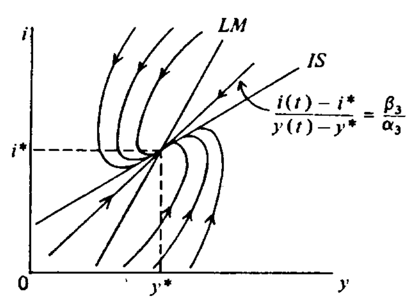

Fig. 3-2

Fig. 3-1. $I_y < S_y$, $\Delta = 0$, $I_i = L_y = 0$ Fig. 3-2. $I_y > S_y$, $\Delta = 0$, *IS* slope < *LM* slope

Case 3 (a) (Star-shaped) stable node ($\lambda_1 = \lambda_2 = \lambda < 0$)

Case 3(b) $\lambda_1 = \lambda_2 = \lambda > 0$. The conditions for both equal roots to be positive to appear require that $I_y > S_y$, $\Delta = 0$, and LM be steeper than IS. The orbits are the same aş those in Fig. 3-2 with the arrows of motion reversed. Since all solutions spproach ∞ as t tends to ∞, the equilibrium point is *an unstable node* which iş depicted in Fig. 3-3.

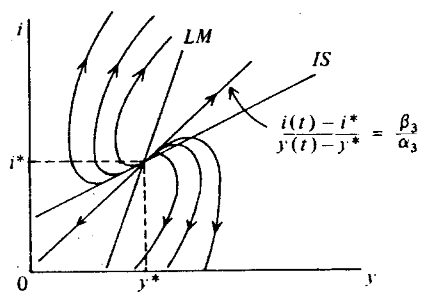

Fig. 3-3

Fig. 3-3. $I_y > S_y$, $\Delta=0$, *IS* slope<*LM* slope
Cose 3(b) unstable node ($\lambda_1 = \lambda_2 = \lambda > 0$)

Case 4 λ_1 *and* λ_2 *are complex conjugates but not pure imaginary.*

The complex conjugates roots are assumed to be $\lambda_1 = a+ib$, and $\lambda_2 = a-ib$ with $a \neq 0$, $b \neq 0$. Each solution pair is:

$$y(t) = y^* + Ae^{at}\cos(bt+\alpha_1),\ A \neq 0,\ 0 < \alpha_1 < 2\pi \tag{23-1}$$

$$i(t) = i^* + Be^{at}\cos(bt+\alpha_2),\ B \neq 0,\ 0 < \alpha_2 < 2\pi \tag{23-2}$$

with

$$\frac{i(t)-i^*}{y(t)-y^*} = \frac{B\cos(bt+\alpha_2)}{A\cos(bt+\alpha_1)},\ \cos(bt+\alpha_1) \neq 0$$

Since this expression is periodic, the ratio $\frac{i(t)-i^*}{y(t)-y^*}$ does not approach a limit as t tends to ∞, but the orbits must circle around the equilibrium point (y*, i*). We may obtain *a stable focus* or *spiral point* as $a<0$, and an *unstable focus as* $a>0$.

The conditions for this appearance require that the coefficients in equation (8) satisfy that 1. LM be steeper than IS, 2. $\Delta<0$, 3. IS may slope either downward ($I_y < S_y$) or upward ($I_y > S_y$) as $a<0$, and it should slope upward as $a>0$, and 4. For

$I_y > S_y$, the relative speed of dynamic adjustment, $\frac{k_2}{k_1}$, should be larger than $\frac{I_y - S_y}{-L_i}$ if a<0 (stable), and less than $\frac{I_y - S_y}{-L_i}$ if a>0 (unstable). The phase diagrams for the case of positive IS slope with a<0 and a>0 are depicted in Fig. 4-1 and 4-2.

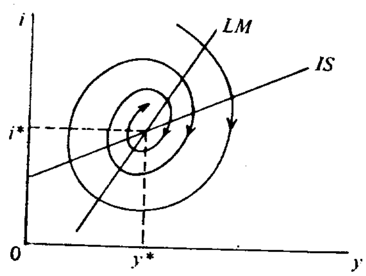

Fig. 4-1

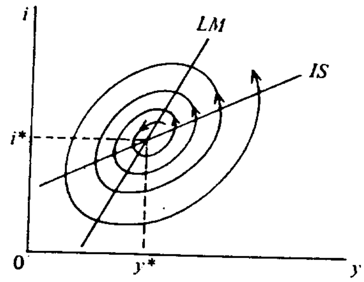

Fig. 4-2

Fig. 4-1. $a<0, \Delta<0, I_y>S_y, \frac{k_2}{k_1} > \frac{I_y - S_y}{-L_i}$ Fig. 4-2. $a>0, \Delta<0, I_y>S_y, \frac{k_2}{k_1} < \frac{I_y - S_y}{-L_i}$

(stable and unstable focus)

Case 5 λ_1 and λ_2 are pure imaginary. Now the characteristic roots will be λ_1 = bi and λ_2 = −bi, and the solution (23) with a=0 then becomes

$$y(t) = y^* + A\cos(bt+\alpha_1) \tag{24-1}$$

$$i(t) = i^* + B\cos(bt+\alpha_2) \tag{24-2}$$

Clearly y(t) and i(t) are periodic functions with period $\frac{2\pi}{b}$ so that every orbit beginning at the point $(y^*+\hat{y}, i^*+\hat{i})$ when $t=\hat{t}$ will return to the same point when $t=\hat{t}+\frac{2\pi}{b}$ as

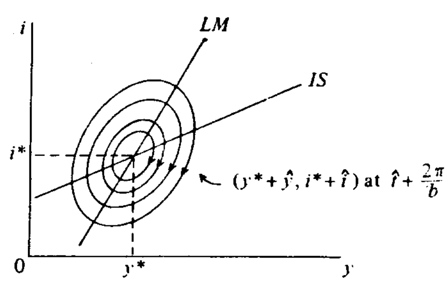

Fig. 5

Fig. 5. $\Delta<0, I_y>S_y, \frac{k_2}{k_1} = \frac{I_y - S_y}{-L_i}$

(Center)

shown in Fig. 5. The orbits are closed curves and the equilibrium point (y*, i*) is *a stable center but not asymptotically stable,* since the general solutions will not approach (y*, i*). The conditions foı this case to happen requıre that 1. $\Delta<0$, 2. IS slope upwards ($I_y > S_y$), 3. LM be steeper than IS, and 4. Relative adjustment ratio, $\frac{k_2}{k_1}$, be just equal to the critical point, $\frac{I_y - S_y}{-L_i}$.

IV. Heuristic Conclusion

We have already examined and clarified the following five major types of equilibrium point in a widely used Keynesian IS-LM framework: 1. Stable node and unstable node, 2. saddle point, 3. starshaped stable node and unstable node, 4. stable focus and unstable focus, and 5. center.

It is usually wrong and misleading for the aforementioned popular modern macroeconomics textbooks to regard the conventional IS-LM model as dynamically stable in case that IS is positively sloped and when it is flatter than LM. As this paper has already shown, that case may encounter *unstable node* if $\lambda_1>\lambda_2>0$ and $\Delta>0$ or if $\lambda_1=\lambda_2=\lambda=0$ and $\Delta=0$, *unstable focus* if $\frac{k_2}{k_1}<\frac{I_y-S_y}{-L_i}$ and $\Delta<0$, and *asymptotically unstable center* if $\frac{k_2}{k_1}=\frac{I_y-S_y}{-L_i}$ and $\Delta<0$.

Our paper indeed renders a piece of worthwhile information for the pedagogical clarification of the long-existing misunderstanding of the IS-LM dynamic stability analysis. This demonstrated our heuristic purpose.

Footnote

1. The solutions pairs $\{\alpha_1 e^{(a+ib)t}, \beta_1 e^{(a+ib)t}\}$ *and* $\{\alpha_2 e^{(a-ib)t}, \beta_2 e^{(a-ib)t}\}$ are linearly independent. The constants α_1, β_1, α_2 and β_2 are complex numbers. To obtain real solution pairs, letting $\alpha_1 = A_1 + iA_2$, $\beta=\beta_1+i\beta_2$ and applying Euler's formula, $e^{i\theta}=\cos\theta + i\sin\theta$ to the first complex solution pair, we get

$$y(t) = y^* + (A_1 + iA_2)e^{at}(\cos bt + i\sin bt), \qquad \text{(A1)}$$

$$i(t) = i^* + (B_1 + iB_2)e^{at}(\cos bt + i\sin bt) \qquad \text{(A2)}$$

Remembering that $i^2 = -1$, the equations then become

$$y(t) = y^* + e^{at}[(A_1 \cos bt - A_2 \sin bt) + i(A_1 \sin bt + A_2 \cos bt)] \qquad \text{(A3)}$$

$$i(t) = i^{*}+e^{at}[(B_1 \cos bt - B_2 \sin bt)+i(B_1 \sin bt + B_2 \cos bt)] \qquad \text{(A4)}$$

Since the coefficients of the system (7) are real, the only way $[y, i]$ can be a solution pair is for all the real terms and similarly the imaginary terms to cancel out. Thus the real parts of y and i must form a solution pair of the system (7):

$$y(t) = y^{*}+e^{at}[c_1(A_1 \cos bt - A_2 \sin bt)+c_2(A_1 \sin bt + A_2(\cos bt)], \qquad \text{(A5)}$$

$$i(t) = i^{*}+e^{at}[c_1(B_1 \cos bt - B_2 \sin bt)+c_2(B_1 \sin bt + B_2 \cos bt)] \qquad \text{(A6)}$$

To simplify the notation, we define

$$k_1 = c_1A_1+c_2A_2, \quad k_2 = -c_1A_2+c_2A_1,$$

$$k_3 = c_1B_1+c_2B_2, \quad k_4 = -c_1B_2+c_2B_1$$

Equations (A5) and (A6) then become

$$y(t) = y^{*}+e^{at}(k_1 \cos bt + k_2 \sin bt), \qquad \text{(A7)}$$

$$i(t) = i^{*}+e^{at}(k_3 \cos bt + k_4 \sin bt) \qquad \text{(A8)}$$

Equations (A7) and (A8) can be rearranged as

$$y(t) = y^{*}+A(\frac{k_1}{A} \cos bt + \frac{k_2}{A} \sin bt)e^{at}, \qquad \text{(A9)}$$

$$i(t) = i^{*}+B(\frac{k_3}{B} \cos bt + \frac{k_4}{B} \sin bt)e^{at} \qquad \text{(A10)}$$

with $A = \sqrt{k_1^2+k_2^2}$ and $B = \sqrt{k_3^2+k_4^2}$.

By the appropriate choice of A and B, $\frac{k_1}{A}$, $\frac{k_2}{A}$, $\frac{k_3}{B}$, and $\frac{k_4}{B}$ will all lie between -1 and +1. Thus we may define α_1 and α_2 by

$$\cos\alpha_1 = \frac{k_1}{A}, \cos\alpha_2 = \frac{k_3}{B}, \sin\alpha_1 = \frac{-k_2}{A}, \sin\alpha_2 = \frac{-k_4}{B}, 0<\alpha_1, \alpha_2<2\pi$$

Finally, equations (A9) and (A10) reduce to

$$y(t) = y^{*}+Ae^{at}(\cos\alpha_1 \cos bt - \sin\alpha_1 \sin bt) \qquad \text{(A11)}$$

$$= y^* + Ae^{at} \cos(bt + \alpha_1)$$

$$i(t) = i^* + Be^{at}(\cos \alpha_2 \cos bt - \sin \alpha_2 \sin bt) \quad \text{(A12)}$$

$$= i^* + Be^{at} \cos(bt + \alpha_2)$$

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