ON ADMISSIBLE ESTIMATOR AND UNIFORMLY ADMISSIBLE SAMPLING STRATEGY FOR A FINITE POPULATION PROPORTION

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I. INTRODUCTION

Admissible estimation in relation to sampling survey has been studied in great detail by Godambe [2], [4], Godambe and Joshi [3], Joshi [6], [7], [8], Ericson [1], Sekkappan and Thompson [10], and Scott [9]. Godambe [4], Joshi [6], [7], Ericson [1], and Sekkappan and Thompson [10] have established the uniform admissibility of some classes of estimator-design pairs for a finite population total or for a finite population mean. In particular, Joshi [7] showed that the sample mean and a sampling design of fixed sample size n are uniformly admissible for the population mean, when the competing designs have expected sample size not less than n.

In this paper, the admissibility of the sample proportion and the uniform admissibility of the estimator-design pair consisting of the sample proportion and any fixed sample size design are studied.

II. NOTATION AND DEFINITIONS

Let U denote a finite population of N identifiable elements tagged with the labels i = 1, 2, . . . , N, i.e.

$$U = \{1, 2, \ldots, N\}$$

Let A be a subset of U having some characteristic or attribute of interest. Define on U a real variable x as follows:

$$x_i = \begin{cases} 1, & \text{if the i-th unit of U belongs to A} \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Let $\underline{x} = (x_1, x_2, \ldots, x_N)$ and $\mathcal{X}$ be the set of all possible points $\underline{x}$, i.e. $\mathcal{X}$ is a set of the Cartesian product of N sets $\{0, 1\}$, namely,

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$$\mathfrak{X} = \{(x_1, x_2, \ldots, x_N): x_i = 0, 1; i = 1, 2, \ldots, N\} \tag{2}$$

The population proportion of units in A is

$$\theta(\underline{x}) = \sum_{i=1}^{N} x_i/N \tag{3}$$

For estimating the population proportion $\theta(\underline{x})$, a sample s is any subset of U selected according to some sampling design d = (S, p), where S is the sample space consisting of all possible samples s and p is a probability measure defined on S such that

$$\begin{aligned} &\text{(i)} \quad 0 \leqslant p(s) \leqslant 1 \text{ for all } s \in S\,, \\ &\text{(ii)} \quad \sum_{s \in S} p(s) = 1\,. \end{aligned} \tag{4}$$

Every possible design of random sampling is a special case of d = (S, p).

For a sample s, n(s) will denote the number of distinct units in the sample and will be called the sample size of s. A sampling design d = (S, p) is said to be of fixed sample size m if p(s) = 0 whenever n(s) $\neq$ m, a fixed integer.

For estimation of the population proportion $\theta(\underline{x})$, an estimator is defined as follows:

[Definition 1] Any real function $\hat{\theta}$ on the product space S$\times\mathfrak{X}$, such that $\hat{\theta}(s, \underline{x})$ depends on $\underline{x}$ only through those x_i's for which $i \in s$, is called an estimator of $\theta(\underline{x})$.

[Definition 2] For a given sampling design d = (S, p), an estimator $\hat{\theta}(s, \underline{x})$ is admissible for the population proportion $\theta(\underline{x})$, if there exists no other estimator $\tilde{\theta}(s, \underline{x})$ such that

$$E_p[\tilde{\theta}(s, \underline{x}) - \theta(\underline{x})]^2 \leqslant E_p[\hat{\theta}(s, \underline{x}) - \theta(\underline{x})]^2 \tag{5}$$

for all $\underline{x} \in \mathfrak{X}$, and the strict inequality in (5) holds for at least one $\underline{x} \in \mathfrak{X}$.

The two quantities in (5) are mean squared-errors of the two estimators, i.e. the mean squared-error of $\hat{\theta}(s, \underline{x})$ is

$$E_p[\hat{\theta}(s, \underline{x}) - \theta(\underline{x})]^2 = \sum_{s \in S} [\hat{\theta}(s, \underline{x}) - \theta(\underline{x})]^2\, p(s)$$

[Definition 3] A pair $(\hat{\theta}, p)$ consisting of an estimator $\hat{\theta}(s, \underline{x})$ and a sampling design d = (S, p) is called a sampling strategy for estimating $\theta(\underline{x})$.

[Definition 4] A sampling strategy $(\hat{\theta}, p)$ is said to be uniformly admissible for $\theta(\underline{x})$, if there exists no other sampling strategy $(\tilde{\theta}, \tilde{p})$ such that

$$\text{(i)}\quad E_{\tilde{p}}\,[\,n(s)\,] \leqslant E_p\,[n(s)]\,, \tag{6}$$

$$\text{(ii)}\quad E_{\tilde{p}}\,[\,\tilde{\theta}(s, \underline{x}) - \theta(\underline{x})\,]^2 \leqslant E_p\,[\hat{\theta}(s, \underline{x}) - \theta(\underline{x})\,]^2 \tag{7}$$

for all $\underline{x} \in \mathbf{X}$, and the strict inequality holds either in (6) or for at least one $\underline{x} \in \mathbf{X}$ in (7).

The definitions given above follow those of Godambe [2], Godambe and Joshi [3], etc. Joshi [6] showed that in the entire class of all estimators, linear and non-linear, biased and unbiased, the sample mean is always admissible as estimator of the population mean on the N-dimensional Euclidean space R^N or on any subset of R^N given by

$$\{(x_1, x_2, \ldots, x_N): c_1 \leqslant x_i \leqslant c_2,\ i = 1, 2, \ldots, N\}$$

where c_1 and c_2 are some arbitrary constants. In this paper, the sample proportion will be proved to be admissible for the population proportion on the space $\mathbf{X}$.

III. ADMISSIBILITY OF THE SAMPLE PROPORTION

Let an estimator $\hat{\theta}(s, \underline{x})$ of the finite population proportion $\theta(\underline{x})$ of units in A be defined as follows:

$$\hat{\theta}(s, \underline{x}) = \sum_{i \in s} x_i / n(s), \text{ for } \underline{x} \in \mathbf{X} \tag{8}$$

which is the sample proportion of units in A in the sample s of size n(s).

Since the value of x_i on the i-th unit of U is unknown but can be determined when the i-th unit is surveyed, x_i is a random variable. The distribution of x_i is assumed to be Bernoulli distribution with probability function given by

$$f(x; \theta) = \theta^x (1 - \theta)^{1-x}\, I_{\{0,1\}}(x)$$

where $I_A(x)$ is an indicator function of the set A. Furthermore, since all units in U are independent, $x_1, x_2, \ldots, x_N$ are independent random variables having the same Bernoulli distribution. The distribution of $x_1, x_2, \ldots, x_N$ will be considered as aprior distribution in the proof of admissibility of the sample proportion.

[Lemma 1] Let $X_1, X_2, \ldots, X_m$ be independent random variables having the same Bernoulli distribution $b(1, \theta)$. Let $\hat{\theta}(X_1, X_2, \ldots, X_m)$ be a statistic and let $T = X_1 + X_2 + \ldots + X_m$, then

$$\int_0^1 [\hat{\theta}(x_1, x_2, \ldots, x_m) - \theta]^2 \theta^{t-1}(1-\theta)^{m-t-1} d\theta \tag{9}$$

is minimum only when $\hat{\theta}(X_1, X_2, \ldots, X_m) = \sum_{i=1}^{m} X_i/m$ with probability one.

Proof: Let $\underline{x} = (x_1, x_2, \ldots, x_m)$. If the quantity in (9) is divided by the beta function $B(t, m\text{-}t) = \Gamma(t)\,\Gamma(m\text{-}t)/\Gamma(m)$, then it becomes

$$\begin{aligned} &\frac{1}{B(t, m\text{-}t)} \int_0^1 [\theta - \hat{\theta}(\underline{x})]^2 \theta^{t-1}(1-\theta)^{m-t-1} d\theta \\ &= E\,[\Theta - \hat{\theta}(\underline{x})\,]^2 \end{aligned} \tag{10}$$

where Θ is a random variable having beta distribution with parameters t and m-t. The mean and variance of Θ are respectively given by

$$E(\Theta) = \frac{t}{m}, \quad \text{Var}\,(\Theta) = \frac{t(m-t)}{m^2(m+1)} \tag{11}$$

For all $\underline{x} \in \{\underline{x}: \sum_{i=1}^{m} x_i = t, 0 < t < m\}$, it is obvious that the following equality holds:

$$E\,[\Theta - \hat{\theta}(\underline{x})]^2 = \text{Var}(\Theta) + [E(\Theta) - \hat{\theta}(\underline{x})]^2 \tag{12}$$

which is minimized only when

$$\hat{\theta}(\underline{x}) = E(\Theta) = \frac{t}{m} = \sum_{i=1}^{m} x_i/m \tag{13}$$

For $\underline{x} = (0, 0, \ldots, 0)$ or $x = (1, 1, \ldots, 1)$ such that $\sum_{i=1}^{m} x_i = t = 0$ or m, we have from (11)

$$\text{Var}\ (\Theta) = 0$$

which implies $P[\Theta = E(\Theta)] = 1$, and thus, $P[\hat{\theta}(x) = E(\Theta)] = 1$.

Hence, we have from (13)

$$\hat{\theta}(\underline{x}) = \left\{ \begin{array}{l} 0 \text{ if } \underline{x} = (0, 0, \ldots, 0) \\ 1 \text{ if } \underline{x} = (1, 1, \ldots, 1). \end{array} \right\}$$

Therefore, for all $\underline{x} \in \{\underline{x}\colon x_i = 0, 1; i = i, 2, \ldots, m\}$, the quantity in (9) is minimized only when $\hat{\theta}(\underline{x}) = \sum_{i=1}^{m} x_i/m$ a.e.

Q.E.D.

Now we proceed to show the admissibility of the estimator $\hat{\theta}(s, \underline{x})$ given in (8) for the finite population proportion $\theta(\underline{x})$ in the following theorem.

[Theorem 1] For any sampling design d = (S, p), the estimator $\hat{\theta}(s, \underline{x})$ given in (8) is admissible for the finite population proportion $\theta(\underline{x})$.

Proof: If $\hat{\theta}(s, \underline{x})$ is not admissible for $\theta(\underline{x})$, then by definition 2 there exists an estimator $\tilde{\theta}(s, \underline{x})$ such that, for all $x \in \mathfrak{X}$,

$$\sum_{s \in S} p(s)[\tilde{\theta}(s, \underline{x}) - \theta(x)]^2 \leqslant \sum_{s \in S} p(s)[\hat{\theta}(s, \underline{x}) - \theta(\underline{x})]^2 \tag{14}$$

For the sample s with sample size n(s) = N, i.e. s = U, it is enough to consider in (14) estimator $\tilde{\theta}$ such that $\tilde{\theta}(U, \underline{x}) = \theta(\underline{x})$. Now, for a sample s with n(s) $<$ N. let

$$g(s, \underline{x}) = [N\tilde{\theta}(s, \underline{x}) - \sum_{i \in s} x_i]/[N - n(s)] \tag{15}$$

Let S* = S – U and rewrite (14) as follows:

$$\sum_{s \in S^*} p(s) [N - n(s)]^2 [g(s, \underline{x}) - h(s, \underline{x})]^2$$
$$\leqslant \sum_{s \in S^*} p(s) [N - n(s)]^2 [\hat{\theta}(s, \underline{x}) - h(s, \underline{x})]^2 \tag{16}$$

where $h(s, \underline{x}) = \sum_{i \notin s} x_i/[N - n(s)]$.

Now taking expectation of both sides of (16) with respect to a prior distribution of $x_1, x_2, \ldots, x_N$ over $\mathfrak{X}$ such that $x_1, x_2, \ldots, x_N$ are independently and identically distributed as Bernoulli distribution $b(1, \theta)$, we have

$$\sum_{s \in S^*} p(s) [N - n(s)]^2 E_b [\{g(s, \underline{x}) - \theta\} + \{\theta - h(s, \underline{x})\}]^2$$
$$\leqslant \sum_{s \in S^*} p(s) [N - n(s)]^2 E_b [\{\hat{\theta}(s, \underline{x}) - \theta\} + \{\theta - h(s, \underline{x})\}]^2 \tag{17}$$

Noting that the expected values of the product terms on both sides of (17) vanish due to the independence of x_i's and cancelling out the common term $\sum_{s\in S^*} p(s)\,[N - n(s)]^2\, E_b\,[h(s, \underline{x}) - \theta\,]^2$ on both sides of (17), we have

$$\sum_{s\in S^*} p(s)\,[N - n(s)]^2\, E_b\,[g(s, \underline{x}) - \theta\,]^2$$

$$\leqslant \sum_{s\in S^*} p(s)\,[N - n(s)]^2\, E_b\,[\hat{\theta}(s, \underline{x}) - \theta\,]^2 \tag{18}$$

where $E_b\,[g(s, \underline{x}) - \theta\,]^2 = \sum_{\underline{x}\in \mathcal{X}} [g(s, \underline{x}) - \theta\,]^2\, \theta^{\sum_{i\in s} x_i} (1 - \theta)^{n(s) - \sum_{i\in s} x_i}$

and $\quad E_b\,[\hat{\theta}(s, \underline{x}) - \theta\,]^2 = \sum_{\underline{x}\in \mathcal{X}} [\hat{\theta}(s, \underline{x}) - \theta\,]^2\, \theta^{\sum_{i\in s} x_i} (1 - \theta)^{n(s) - \sum_{i\in s} x_i}$

On both sides of (18) are multiplied by $1/\theta(1\text{-}\theta)$ and then integrated with respect to θ from 0 to 1, we get

$$\sum_{s\in S^*} p(s)[N - n(s)]^2 \sum_{\underline{x}\in \mathcal{X}} T_g(s, \underline{x})$$

$$\leqslant \sum_{s\in S^*} p(s)\,[N - n(s)]^2 \sum_{\underline{x}\in \mathcal{X}} T_{\hat{\theta}}(s, \underline{x}) \tag{19}$$

where $T_g(s, \underline{x}) = \int_0^1 [g\,(s, \underline{x}) - \theta\,]^2\, \theta^{\sum_{i\in s} x_i - 1} (1 - \theta)^{n(s) - \sum_{i\in s} x_i - 1}\, d\theta$

and similarly $T_{\hat{\theta}}(s, \underline{x})$ is defined.
For all s∈S, we have from Lemma 1

$$T_g\,(s, \underline{x}) \geqslant T_{\hat{\theta}}\,(s, \underline{x}) \tag{20}$$

and the equality in (20) holds only when $g(s, \underline{x}) = \hat{\theta}(s, \underline{x})$. Now summing up over all $x \in \mathcal{X}$ on both sides of (20), we have

$$\sum_{x\in \mathcal{X}} T_g\,(s, \underline{x}) \geqslant \sum_{\underline{x}\in \mathcal{X}} T_{\hat{\theta}}\,(s, \underline{x}) \tag{21}$$

On both sides of (21) are multiplied by $p(s)[N - n(s)]^2$ and then summed up over all $s \in S^*$, we get by comparison of the result with (19) that only the equality holds in (19), i.e.

$$\sum_{s \in S^*} p(s)\,[N - n(s)]^2 \sum_{\underline{x} \in \mathcal{X}} T_g(s, \underline{x})$$

$$= \sum_{s \in S^*} p(s)\,[N - n(s)]^2 \sum_{\underline{x} \in \mathcal{X}} T_{\hat{\theta}}(s, \underline{x}) \tag{22}$$

Since both $T_g(s, \underline{x})$ and $T_{\hat{\theta}}(s, \underline{x})$ in (22) are non-negative, we have, for all $\underline{x} \in \mathcal{X}$ and for all $s \in S^*$,

$$T_g(s, \underline{x}) = T_{\hat{\theta}}(s, \underline{x})$$

which implies

$$g(s, \underline{x}) = \hat{\theta}(s, \underline{x}) \quad \text{for all } \underline{x} \in \mathcal{X} \text{ and all } s \in S^*. \tag{23}$$

Substituting (23) in (15), we have

$$\tilde{\theta}(s, \underline{x}) = \hat{\theta}(s, \underline{x}) \text{ for all } \underline{x} \in \mathcal{X} \text{ and all } s \in S^*.$$

Further, for the sample $s = U$, we have

$$\hat{\theta}(U, \underline{x}) = \theta(\underline{x}) = \tilde{\theta}(U, \underline{x}) \quad \text{for all } \underline{x} \in \mathcal{X}.$$

Hence, $\quad \tilde{\theta}(s, \underline{x}) = \hat{\theta}(s, \underline{x})$ for all $\underline{x} \in \mathcal{X}$ and all $s \in S$.

Therefore, the strict inequality in (14) can't hold. This completes the proof of the theorem. Q.E.D.

IV. UNIFORM ADMISSIBILITY

For a given design $d = (S, p)$, the sample proportion $\hat{\theta}(s, \underline{x})$ is shown in Theorem 1 to be admissible for the finite population proportion $\theta(\underline{x})$. In this section, we will find a class of designs in which the estimator $\hat{\theta}(s, \underline{x})$ is uniformly admissible for $\theta(\underline{x})$. The two classes of sampling designs usually considered are $C = \{C_m\}$ and $D = \{D_m\}$, where

$$C_m = \{ d = (S, p) \colon \; p(s) = 0 \;\text{ if }\; n(s) \neq m \} \tag{24}$$

and

$$D_m = \{ d = (S, p) \colon \sum_{s \in S} n(s)\, p(s) = m \}, \tag{25}$$

i.e. the class C consists of sampling designs of fixed sample size and the class D contains sampling designs of fixed expected sample size. It is obvious that the class C is a subclass of the class D. With respect to the class D, the uniform admissibility is defined in Definition 4.

For a given sampling design d = (S, p), let π_i denote the inclusion probability for the i-th unit of U, i.e.

$$\pi_i = \sum_{s \ni i} p(s) \tag{26}$$

where $s \ni i$ denotes all samples s having the i-th unit of U. Further, let π_{ij} denote the inclusion probability for both the i-th and j-th units of U, i.e.

$$\pi_{ij} = \sum_{s \ni i,j} p(s) \tag{27}$$

[Lemma 2] For a given sampling design d = (S, p), the following two equations hold:

$$(1) \sum_{i=1}^{N} \pi_i = E_p[n(s)] \tag{28}$$

$$(2) \sum_{i<j}^{N} \pi_{ij} = E_p\left[\binom{n(s)}{2}\right] \tag{29}$$

Proof:

$$\sum_{i=1}^{N} \pi_i = \sum_{i=1}^{N} \sum_{s \ni i} p(s) = \sum_{s \in S} \sum_{i \in s} p(s) = \sum_{s \in S} n(s)\, p(s) = E_p[n(s)].$$

$$\sum_{i<j}^{N} \pi_{ij} = \sum_{i<j}^{N} \sum_{s \ni i,j} p(s) = \sum_{s \in S} \sum_{\substack{i<j \\ i,j \in s}} p(s) = \sum_{s \in S} \binom{n(s)}{2} p(s)$$

$$= E_p\left[\binom{n(s)}{2}\right].$$ Q.E.D.

The relationship between the inclusion probabilities π_i's and the sample sizes n(s)'s is thus clear from (28) that as soon as the inclusion probabilities π_i's are specified the expected sample size $E_p[n(s)]$ is automatically fixed for the sampling design d = (S, p).

[Theorem 2] Let a sampling design d = (S, p) belong to the class $C = \{C_m\}$ defined in (24) and let $\hat{\theta}(s, \underline{x})$ be the sample proportion, i.e.

$$\hat{\theta}(s, \underline{x}) = \sum_{i \in s} x_i / n(s), \quad \underline{x} \in \mathcal{X}$$

then the sampling strategy $(\hat{\theta}, p)$ is uniformly admissible among sampling strategies (θ^*, p^*), where $d^* = (S^*, p^*) \in D$ defined in (25).

Proof: If the sampling strategy $(\hat{\theta}, p)$ is not uniformly admissible for the finite population proportion $\theta(\underline{x})$, then there exists a sampling strategy (θ^*, p^*) such that

$$E_{p^*}[n^*(s)] \leqslant E_p[n(s)] = m \tag{30}$$

and

$$E_{p^*}[\theta^*(s, \underline{x}) - \theta(\underline{x})]^2 \leqslant E_p[\hat{\theta}(s, \underline{x}) - \theta(\underline{x})]^2 \tag{31}$$

for all $x \in \mathfrak{X}$, the strict inequality holds either in (30) or for at least one $x \in \mathfrak{X}$ in (31). Define $g^*(s, \underline{x})$ on $S^* \times \mathfrak{X}$ as follows:

$$N\theta^*(s, \underline{x}) = [N - n^*(s)]\ g^*(s, \underline{x}) + \sum_{i \in s} x_i \tag{32}$$

and let $S_p = \{ s \in S: p(s) > 0 \}$ and $S_{p^*} = \{ s \in S^*: p^*(s) > 0 \}$, then substituting (32) in (31), we have, for all $\underline{x} \in \mathfrak{X}$,

$$\sum_{s \in S_{p^*}} p^*(s)[N - n^*(s)]^2\ [g^*(s, \underline{x}) - h^*(s, \underline{x})]^2$$
$$\leqslant \sum_{s \in S_p} p(s)\ (N - m)^2\ [\hat{\theta}(s, \underline{x}) - h(s, \underline{x})]^2 \tag{33}$$

where $h^*(s, \underline{x}) = \sum_{i \notin s} x_i / [N - n^*(s)]$ for $s \in S_{p^*}$

and $h(s, \underline{x}) = \sum_{i \notin s} x_i / (N - m)$ for $s \in S_p$.

Now, taking the expectations of both sides of (33) with respect to a prior distribution of $x_1, x_2, \ldots, x_N$ on $\mathfrak{X}$, under which all the x_i $(i = 1, 2, \ldots, N)$ are independently and identically distributed as Bernoulli distribution $b(1, \theta)$, we have

$$\sum_{s \in S_{p^*}} p^*(s)[N - n^*(s)]^2\ E_b[g^*(s, \underline{x}) - h^*(s, \underline{x})]^2$$
$$\leqslant \sum_{s \in S_p} p(s)\ (N - m)^2\ E_b[\hat{\theta}(s, \underline{x}) - h(s, \underline{x})]^2 \tag{34}$$

where, for $s \in S_{p^*}$,

$$E_b[g^*(s, \underline{x}) - h^*(s, \underline{x})]^2$$

$$= E_b[g^*(s, \underline{x}) - \theta]^2 + E_b[h^*(s, \underline{x}) - \theta]^2$$

$$= E_b[g^*(s, \underline{x}) - \theta]^2 + \theta(1 - \theta)/[N - n^*(s)]$$

and, for $s \in S_p$,

$$E_b[\hat{\theta}(s, \underline{x}) - h(s, \underline{x})]^2 = E_b[\hat{\theta}(s, \underline{x}) - \theta]^2 + \theta(1-\theta)/(N - m)$$

Thus, (34) becomes

$$\sum_{s \in S_{p^*}} p^*(s)[N - n^*(s)]^2 E_b[g^*(s, \underline{x}) - \theta]^2 + \theta(1-\theta) \sum_{s \in S_{p^*}} p^*(s)[N - n^*(s)]$$

$$\leqslant \sum_{s \in S_p} p(s)(N - m)^2 E_b[\hat{\theta}(s, \underline{x}) - \theta]^2 + \theta(1-\theta) \sum_{s \in S_p} p(s)(N - m)$$

Since $\sum_{s \in S_{p^*}} p^*(s) = \sum_{s \in S_p} p(s) = 1$, we have, after cancelling out the common term $N\theta(1 - \theta)$ on both sides of the above inequality,

$$\sum_{s \in S_{p^*}} p^*(s)[N - n^*(s)]^2 E_b[g^*(s, \underline{x}) - \underline{\theta}]^2 - \theta(1-\theta) E_{p^*}[n^*(s)]$$

$$\leqslant \sum_{s \in S_p} p(s)(N - m)^2 E_b[\hat{\theta}(s, \underline{x}) - \theta]^2 - \theta(1-\theta)m \qquad (35)$$

The expectations on both sides of (35) can be expressed as follows:

$$E_b[g^*(s, \underline{x}) - \theta]^2$$

$$= \sum_{\underline{x} \in \mathcal{X}} [g^*(s, \underline{x}) - \theta]^2 \theta^{\sum_{i \in s} x_i} (1 - \theta)^{n^*(s) - \sum_{i \in s} x_i}$$

$$E_b[\hat{\theta}(s, \underline{x}) - \theta]^2$$

$$= \sum_{\underline{x} \in \mathcal{X}} [\hat{\theta}(s, \underline{x}) - \theta]^2 \theta^{\sum_{i \in s} x_i} (1 - \theta)^{m - \sum_{i \in s} x_i}$$

On both sides of (35) are divided by $\theta(1-\theta)$ and then integrated with respect to θ from 0 to 1, we get

$$\sum_{s \in S_{p^*}} p^*(s)[N - n^*(s)]^2 \sum_{\underline{x} \in \mathcal{X}} T_{g^*}(s, \underline{x}) - E_{p^*}[n^*(s)]$$

$$\leqslant \sum_{s\in S_p} p(s)\,(N-m)^2 \sum_{\underline{x}\in \mathcal{X}} T_{\hat\theta}(s, \underline{x}) - m \tag{36}$$

where $$T_{g*}(s, \underline{x}) = \int_0^1 [g^*(s, \underline{x}) - \theta]^2\, \theta^{\sum_{i\in s} x_i - 1} (1-\theta)^{n^*(s) - \sum_{i\in s} x_i - 1}\, d\theta$$

and $T_{\hat\theta}(s, \underline{x})$ is similarly defined.
If in the sampling design $d^* = (S^*, p^*)$, we also take the sample proportion as an estimator for the finite population proportion, i.e.

$$\hat\theta^*(s, \underline{x}) = \sum_{i\in s} x_i / n^*(s),$$

then we have from Lemma 1

$$T_{\hat\theta*}(s, \underline{x}) \leqslant T_{g*}(s, \underline{x}) \tag{37}$$

for all $x \in \mathcal{X}$ and for all $s \in S^*$, and the equality in (37) holds only when $\hat\theta^*(s, \underline{x}) = g^*(s, \underline{x})$. From (36) and (37), we have

$$\sum_{s\in S_{p*}} p^*(s)[N - n^*(s)]^2 \sum_{\underline{x}\in \mathcal{X}} T_{\hat\theta*}(s, \underline{x}) - E_{p*}[n^*(s)]$$

$$\leqslant \sum_{s\in S_p} p(s)\,(N-m)^2 \sum_{\underline{x}\in \mathcal{X}} T_{\hat\theta}(s, \underline{x}) - m \tag{38}$$

The left hand side (LHS) of (38) can be computed as follows:

$$\text{LHS} = \sum_{s\in S_{p*}} p^*(s)[N - n^*(s)]^2 \int_0^1 E_b\,[\hat\theta^*(s, \underline{x}) - \theta]^2 \frac{1}{\theta(1-\theta)}\, d\theta - E_{p*}[n^*(s)]$$

$$= \sum_{s\in S_{p*}} p^*(s)[N - n^*(s)]^2 \int_0^1 \frac{\theta(1-\theta)}{n^*(s)} \cdot \frac{1}{\theta(1-\theta)}\, d\theta - E_{p*}[n^*(s)]$$

$$= N^2 E_{p*}\left[\frac{1}{n^*(s)}\right] - 2N$$

By similar computation of the right hand side (RHS) of (38), we obtain

$$\text{RHS} = \sum_{s\in S_p} p(s)(N-m)^2 \int_0^1 E_b\,[\hat\theta(s, \underline{x}) - \theta]^2 \frac{1}{\theta(1-\theta)}\, d\theta - m$$

$$= \frac{N^2}{m} - 2N$$

Thus, (38) becomes

$$N^2 E_{p^*}[\frac{1}{n^*(s)}] - 2N \leqslant \frac{N^2}{m} - 2N$$

or $$E_{p^*}[\frac{1}{n^*(s)}] \leqslant \frac{1}{m} \tag{39}$$

But, it follows from (30) that

$$E_{p^*}[\frac{1}{n^*(s)}] \geqslant \frac{1}{m} \tag{40}$$

From (39) and (40), we have

$$E_{p^*}[\frac{1}{n^*(s)}] = \frac{1}{m} \tag{41}$$

Hence, $n^*(s) = m$ for all $s \in S_{p^*}$. It implies that the sampling design $d^* = (S^*, p^*)$ is also of fixed sample size m. Therefore, the strict inequality in (30) can't hold.

We shall next show that the strict inequality in (31) can't hold. Since $n^*(s) = m$ for all $s \in S_{p^*}$, the equality in (38) holds. Thus, the equality in (37) holds too, and hence

$$g^*(s, \underline{x}) = \hat{\theta}^*(s, \underline{x}) \text{ for all } s \in S_{p^*} \text{ and all } \underline{x} \in \mathbf{X}.$$

Now, we have from (32) and the above equality

$$\theta^*(s, \underline{x}) = \hat{\theta}^*(s, \underline{x}) = \sum_{i \in s} x_i/m \text{ for all } s \in S_{p^*} \text{ and all } \underline{x} \in \mathbf{X}.$$

Let the inclusion probabilities for the units i ($i = 1, 2, \ldots, N$) and for the pairs of units i and j ($i,j = 1, 2, \ldots, N$) for $d = (S, p)$ and $d^* = (S^*, p^*)$ be given by

$$\pi_i = \sum_{s \ni i} p(s), \qquad \pi_i^* = \sum_{s \ni i} p^*(s)$$

$$\pi_{ij} = \sum_{s \ni i,j} p(s), \qquad \pi_{ij}^* = \sum_{s \ni i,j} p^*(s).$$

It is then easily found that

$$E_p[\hat{\theta}(s, \underline{x}) - \theta(\underline{x})]^2$$
$$= \frac{1}{N^2}\left\{\sum_{i=1}^{N} x_i^2 [\pi_i(\frac{N^2}{m} - 2\frac{N}{m}) + 1] + 2\sum_{i<j} x_i x_j [\pi_{ij}\frac{N^2}{m^2} - \frac{N}{m}(\pi_i + \pi_j) + 1]\right\} \tag{42}$$

Similarly, we have

$$E_{p^*}[\theta^*(s, \underline{x}) - \theta(\underline{x})]^2 = E_{p^*}[\hat{\theta}^*(s, \underline{x}) - \theta(\underline{x})]^2$$

$$= \frac{1}{N^2}\left\{\sum_{i=1} x_i^2 \left[\pi_i^*\left(\frac{N^2}{m} - 2\frac{N}{m}\right) + 1\right] + 2\sum_{i<j} x_i x_j \left[\pi_{ij}^* \frac{N^2}{m^2} - \frac{N}{m}(\pi_i^* + \pi_j^*) + 1\right]\right\} \quad (43)$$

Now (31) clearly implies that the coefficient of x_i^2 for each i, (i = 1, 2, . . . , N) in the right hand side of (43) must be $\leqslant$ the coefficient of x_i^2 in the right hand side of (42) as otherwise (43) will exceed (42) if we put $x_i = 1$ and all $x_j = 0$, $j \neq i$, j = 1, 2, . . . , N. Thus we have from (42) and (43)

$$\pi_i^* \leqslant \pi_i, \; i = 1, 2, \ldots, N$$

then $$\sum_{i=1}^{N} \pi_i^* \leqslant \sum_{i=1}^{N} \pi_i$$

But, from Lemma 2, we have

$$\sum_{i=1}^{N} \pi_i^* = \sum_{i=1}^{N} \pi_i = m$$

Hence $$\pi_i^* = \pi_i, \; i = 1, 2, \ldots, N \quad (44)$$

Next, if we put $x_i = x_j = 0$ and all $x_k = 0$, $k \neq i, j$, then in both (42) and (43) all coefficients other than those of the terms x_i^2, x_j^2 and $2x_i x_j$ vanish, since by (44) the coefficients of the terms x_i^2 and x_j^2 are equal, we have

$$\pi_{ij}^* \leqslant \pi_{ij}, \text{ for all } i < j.$$

then $$\sum_{i<j}^{N} \pi_{ij}^* \leqslant \sum_{i<j}^{N} \pi_{ij}$$

But, from Lemma 2, we have

$$\sum_{i<j}^{N} \pi_{ij}^* = \sum_{i<j}^{N} \pi_{ij} = \binom{m}{2}$$

Hence $$\pi_{ij}^* = \pi_{ij}, \text{ for all } i < j. \quad (45)$$

It now follows from (42), (43), (44) and (45) that in (31) we have

$$E_{p^*}[\theta^*(s, \underline{x}) - \theta(\underline{x})]^2 = E_p[\hat{\theta}(s, \underline{x}) - \theta(\underline{x})]^2$$

for all $\underline{x} \in \mathfrak{X}$ and the strict inequality in (31) does not hold for any $\underline{x} \in \mathfrak{X}$. The theorem is thus proved. Q.E.D.

V. CONCLUSION AND SUMMARY

There are two classes of sampling designs usually considered in survey-sampling, namely $C = \{C_m\}$ and $D = \{D_m\}$, where

$$C_m = \{d = (S, p): p(s) = 0 \text{ if } n(s) \neq m\}$$

and
$$D_m = \{d = (S, p): \sum_{s \in S} n(s)\, p(s) = m\}.$$

If a sampling design $d = (S, p)$ belongs to the class C, and if $\underline{x} = (x_1, x_2, \ldots, x_N)$ is an element of R^N, the N-dimensional Euclidean space, and let $\tilde{\theta}(s, \underline{x})$ be an estimator of the finite population mean $\theta(\underline{x}) = \sum_{i=1}^{N} x_i/N$ given by

$$\tilde{\theta}(s, \underline{x}) = \frac{1}{N} \sum_{i \in s} b_i x_i$$

where (i) $b_i \geqslant 1$, $i = 1, 2, \ldots, N$ and (ii) $\sum_{i=1}^{N} b_i^{-1} = m$, then Joshi, V. M. (1965) showed that $\tilde{\theta}(s, \underline{x})$ is admissible for $\theta(\underline{x})$ in the class C for almost all $\underline{x} \in R^N$ (Lebesgue measure). But if $\underline{x}$ is an element of $\mathfrak{X} = \{\underline{x}: x_i = 0, 1; i = 1, 2, \ldots, N\}$, then the estimator $\tilde{\theta}(s, \underline{x})$ is not necessarily admissible, since $\mathfrak{X}$ is a set of Lebesgue measure zero. For example, consider the artificial population $U = \{1, 2\}$ and define the sampling design $d = (S, p)$ by $p(s_1) = p(s_2) = ½$, $s_i = \{i\}$, $i = 1, 2$. Let

$$\tilde{\theta}(s, \underline{x}) = \sum_{i \in s} b_i x_i, \text{ where } b_1 = 3/2,\ b_2 = 3/4.$$

Then, $\tilde{\theta}(s, \underline{x})$ is admissible for $\theta(\underline{x}) = ½(x_1 + x_2)$ for almost all $\underline{x} \in R^2$. Now, if $\underline{x} \in \mathfrak{X} = \{(x_1, x_2): x_i = 0, 1; i = 1, 2\}$, then

$$E\,[\tilde{\theta}(s, \underline{x}) - \theta(\underline{x})]^2 = \frac{1}{8}(2x_1 - x_2)^2 + \frac{1}{8}(x_1 - 0.5x_2)^2.$$

Let $\theta^*(s, \underline{x})$ be another estimator of $\theta(x)$ given by

$$\theta^*(s, \underline{x}) = \sum_{i \in s} b_i^* x_i, \text{ where } b_1^* = 1,\ b_2^* = 3/4,$$

then
$$E[\theta^*(s, \underline{x}) - \theta(\underline{x})]^2 = \frac{1}{8}(x_1 - x_2)^2 + \frac{1}{8}(x_1 - 0.5x_2)^2.$$

It is obvious that

$$E[\theta^*(s, \underline{x}) - \theta(\underline{x})]^2 \leqslant E[\tilde{\theta}(s, \underline{x}) - \theta(\underline{x})]^2$$

for all $\underline{x} \in \mathfrak{X}$, and the strict inequality holds when $x_1 = 1$. Thus, $\tilde{\theta}(s, \underline{x})$ is not admissible on $\mathfrak{X}$.

In this paper, we have proved that the sample proportion $\hat{\theta}(s, \underline{x})$ is admissible for the finite population proportion $\theta(\underline{x})$ for any sampling design what so ever on $\mathfrak{X}$. Further, we have proved that the sampling strategy $(\hat{\theta}, p)$, where $\hat{\theta}$ is the sample proportion and the sampling design $d = (S, p)$ is of fixed sample size, i.e. $d \in C$, is uniformly admissible among sampling strategies (θ^*, p^*) on $\mathfrak{X}$, where $d^* = (S^*, p^*) \in D$.

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